

Moral Hazard and Sorting in a Market for Partnerships

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Abstract

We examine how equilibrium sorting patterns in a matching market for partnerships are impacted by the presence of bilateral moral hazard in production. We find that this impact is non-trivial, and its direction depends on whether partners' observable types and their efforts are complements or substitutes: when they are complements (i.e. marginal product of effort is increasing in types), moral hazard favors negative sorting. Conversely, when they are substitutes, moral hazard favors positive sorting. This is due to the differential impact of partners' types on the size and frequency of inefficient punishments required to provide incentives in different environments.

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Partnerships are a widespread form of organizing for production. For example, ? reports that about a quarter of all entrepreneurial firms are partnerships, and that they account for roughly half of total output, assets and employment in privately held businesses.¹ A distinguishing feature of a partnership is that the owners who hold the rights to the commonly

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¹These findings are consistent across three different datasets: the Panel Study of Entrepreneurial Dynamics, the Survey of Consumer Finances and the Survey of Small Business Finances. In ?, partnerships are defined as entrepreneurial firms with two or more co-owners.

produced output also typically exert effort to produce it. Effort of an individual in a partnership is often not fully observable or contractible. It is already recognized in the literature that due to the presence of such bilateral moral hazard, the problem of incentive provision in a partnership is distinct from that in the typical principal-agent problem (for example, Holmstrom (1982); Legros and Matsushima (1991)). In this paper we explore the impact of this problem—i.e. that of incentive provision in partnerships—on another economically important aspect of partnerships. Namely, we ask: in an economy where agents of differing productivities form partnerships, how does the need to provide incentives for effort impact the way in which agents in such an economy are sorted? Do high productivity agents partner with other high productivity agents or do they tend to match with lower productivity agents?

The answer to this question may have important implications for empirical as well as policy analysis. For example, if the need to provide incentives for effort impacts equilibrium sorting patterns, ignoring it in econometric analysis may lead to biased estimates of the underlying technology. From the policy perspective, if the effects of moral hazard on sorting patterns (whether it pushes toward negative or positive sorting) differ across different economic environments, policies that impact the ease of incentive provision (wage floors, caps on bonuses etc.) might also have very different effects and, therefore, may be differentially desirable in these settings. In this paper we argue that both of these considerations are relevant because two-sided moral hazard may indeed have a non-trivial effect on sorting outcomes, and the direction of this effect, indeed, varies across different technological environments.

More specifically, it is well known (e.g. Becker (1973)) that, in the absence of any frictions, equilibrium sorting outcomes in an economy with heterogeneous types are determined by the properties of the production technology: when it is strictly supermodular (strictly submodular) in the partners' types, positive (negative) assortative matching obtains in all equilibria. In this paper, we show that the presence of moral hazard may reinforce or weaken (and sometimes even reverse) the effect of existing technological complementarities on sorting patterns. The direction of this effect depends on whether the overall more productive agents are also more productive on the margin. In other words, whether the observable type of an agent (as measured by his overall productivity) is complementary to his unobservable

effort (in which case marginal product of effort is increasing in type) or a substitute for it.

To understand why, first recall that the equilibrium matchings maximize the total surplus across matches.² Consider a partnership which produces a stochastic output—a success or a failure—and the probability of success depends not only on the partners’ types but also on their decision to work or shirk. When individual efforts are not observable, the partnership’s surplus, in general, would suffer from an efficiency loss. Specifically, incentive provision for effort by both partners requires that they agree to implement punishments in the form of some inefficient action when the low output is realized.³ Thus, formally, the expected size of efficiency loss due to moral hazard is the *product* of the frequency with which low output is realized, and the size of the punishment implemented in that case. Therefore, equilibrium matchings—in order to minimize the total cost of incentive provision—would tend to match types leading to high frequency of punishments with the types that require smaller punishments.⁴ On the one hand, the frequency of punishment is associated with the overall productivity of the matched partners, since punishments are administered only when a failure occurs. On the other hand, the size of the punishment is associated with the marginal productivity of the unobservable effort choices of the matched partners; i.e. on how much the partners’ individual deviations from working to shirking impact the total expected output. The higher the marginal product of effort, the less attractive the deviations, and hence the smaller is the required punishment. Thus, if overall more productive agents are less productive on the margin (i.e., effort and type are substitutes), then the size of the punishment is increasing in type, and, to ensure that large punishments occur infrequently, higher types get matched with other higher types; that is, moral hazard favors positive sorting. Conversely, if the overall more productive agents are also more productive on the margin (i.e., effort and type are complements), the size of the punishment is decreasing in

²When technology exhibits super-(sub-)modularity, this surplus is maximized under positive (negative) sorting. This is the well-understood intuition behind Becker (1973).

³In a static model, such inefficient actions take the form of discarding resources or “money burning”; while in the dynamic setting it could represent the possibility of the partnership’s break-up or inefficient effort choices in the future.

⁴Strictly speaking, this is only one of the forces that impact equilibrium matching patterns. This may act for or against the impact of complementarities potentially present in the underlying technology.

type, and therefore moral hazard favors negative sorting.

What do these theoretical observations imply about the formation of real life economic partnerships? A classic example of a technology where type and effort are substitutes is one where ‘type’ represents the partner’s experience, while effort is exerted to acquire supplementary project-specific information. For instance, consider a partnership of business consultants who jointly work on a project proposal for a potential client. Each consultant specializes in his part of the proposal (financial performance, human resources, advertising, etc.), and in the end the client either accepts or rejects an offer without specifying which part of the proposal was crucial for this decision. The latter, together with unobservability of individual efforts, creates an obvious bilateral moral hazard problem. The more extensive the professional experience of each consultant (e.g. his knowledge of similar businesses), the less new information he might gather by studying the details of this particular business – hence experience and effort are substitutes. Our theoretical results imply that moral hazard would encourage highly experienced consultants to form partnerships with other highly experienced consultants. A similar prediction would apply in other types of business relationships where individual experience or knowledge is crucial. Examples are numerous, such as legal services (where each partner specializes in one aspect of a case, and the final verdict by the jury does not specify which of the presented arguments played the major role in their decision) or healthcare services (where two healthcare professionals, such as a dietitian and a trainer, simultaneously attend to a patient, and it is impossible to determine whose effort was primarily responsible for the final outcome of the treatment).

Our findings also suggest that if the partners’ overall productivity is positively related to their marginal productivity (i.e. type and effort are complements), then moral hazard pushes towards negative sorting. Common examples of business partnerships that fit this environment are ones where ‘type’ is the amount of capital or wealth that each partner brings into a joint business, and effort is the amount of (not fully observable) effort or cost needed to operate the capital. For instance, consider two farmers who own different pieces of equipment used at different stages of farming and form a partnership by agreeing to use this equipment when needed on the same field and then share the harvest. The intensity of operating individual pieces of equipment is not fully observable, and, since harvest is

realized only at the end, it is virtually impossible to determine individual effort choices of each farmer. Hence, the bilateral moral hazard problem arises. Obviously, the productivity of each partner's effort is enhanced by the quality of the equipment he owns. Thus our results predict that, to minimize the cost of incentive provision, farmers with high quality machines should partner with the farmers who own low quality machines. Similar types of technologies are also observed in other capital-intensive industries, such as engineering and construction.

The model's prediction of negative sorting in such environments may seem counterfactual. However, the fact that moral hazard here acts in favor of negative sorting does not necessarily imply that negative sorting has to arise as an equilibrium outcome. An additional important factor that would impact the equilibrium sorting patterns in the opposite direction is technological complementarities between the agents' observable types. In fact, one may argue that such complementarities are likely to exist as they are a justification for why partnerships would have formed in the first place. Which of these effects prevails depends on the details of the specific environments. We demonstrate, however, that it is theoretically possible for the impact of moral hazard to reverse Becker (1973)'s prediction of positive sorting when the underlying technology exhibits this type of complementarities.

Our observation that the implication of moral hazard for sorting patterns is in line with that of underlying technological complementarities in some environments and is the opposite of it in others is particularly interesting because it may explain why partnerships as a form of organizing for production arise more frequently in human-capital intensive industries, such as consulting, accounting or legal services, rather than in physical-capital intensive industries, such as manufacturing or agriculture. The reasoning is that, in the presence of complementarity between types in the underlying production technology, organizing in partnerships would have an advantage in those settings where moral hazard favors positive sorting, i.e. in environments where effort and type are substitutes. This follows since in such environments, in contrast to the ones where effort and type are complements, there exists no tradeoff between the sorting patterns that exploit the complementarity between the types and those that minimize the cost of incentive provision.

Finally, we also illustrate that for a given technological environment, organizational struc-

ture has an impact on patterns of sorting, when moral hazard is present. In particular, we contrast our predictions for sorting patterns in a decentralized market for partnerships to the optimal allocation of workers into teams by a profit maximizing firm owner. To do this, we consider a modification of the model developed in recent work by Franco et al. (2008). We find that, under each of the technological environments we consider, the predicted matching patterns in the decentralized partnership market is exactly the opposite of those chosen by a profit maximizing firm owner. The discrepancy is due to the fact that the optimal compensation scheme chosen by the firm owner requires making additional payouts to the workers after the realization of a *success*. This is in sharp contrast to the partnership problem, where the additional punishments are incurred in case of *failure*. In line with the discussion in the above paragraph, the observation that moral hazard has opposing effects in the two organizational structures suggests that non-employee-owned firms may be more likely to arise than partnerships in capital intensive industries, such as manufacturing, which is broadly consistent with empirical evidence.

The impact of moral hazard on sorting patterns was previously analyzed in economic environments different from the one we study here: Thiele and Wambach (1999) and Newman (2007) focus on one-sided moral hazard where entrepreneurs who are heterogeneous in wealth and whose effort is unobservable, are being sorted across projects with different levels of risk. Citanna and Chakraborty (2005) develop a model with two-sided moral hazard, but asymmetric roles of the partners: one of the tasks in the partnership is more effort intensive than the other. They assume that agents are heterogeneous in wealth, and have limited liability. The latter implies that, to facilitate incentive provision, richer agents have to be allocated to more effort intensive tasks, which results in the matching of richer agents with the poorer ones. In contrast, in our paper, the partners' roles are symmetric, and the main mechanism through which moral hazard affects sorting predictions is very different from the one presented in Citanna and Chakraborty (2005).⁵ Recent literature has also demonstrated that frictions other than moral hazard may impact the link between the technology and the sorting patterns: examples include search frictions (e.g. Shimer and Smith (2000)), incomplete information about the agents' types (e.g. Andersen and Smith (forthcoming)), Kaya

⁵We further discuss the connection of our work with Citanna and Chakraborty (2005) in Section 3.2.

(2008)), borrowing constraints (e.g. Fernandez and Gali (1999)) or limited transferability of utility between the partners in the match (e.g. Legros and Newman (2007)).

The rest of this paper is organized as follows. Section 1 describes our modeling environment. Section 2 characterizes the optimal contract for a partnership of any two types. Section 3 derives our main results about matching predictions due to moral hazard. Section 4 illustrates that our results could be extended to a less stylized environment, where, instead of committing to a contract involving money burning, the partners play a repeated game without commitment. Section 5 illustrates that sorting implications of moral hazard may differ across different organizational structures, and Section 6 concludes.

1 Model

We consider a one-to-one matching market for partnerships with N participants. The participants are heterogeneous with types from $\mathcal{M} \subset \mathbb{R}$. We refer to typical elements of \mathcal{M} as m, n . Throughout we assume that types are publicly known.

The interactions between the participants take place in two stages: in the first stage the participants of the market form partnerships (the matching stage), and in the second stage participants who are matched together in a partnership engage in a productive activity (production stage). Clearly, the interaction in the matching stage anticipates the outcome of the production stage. Below, we first describe the details of the production stage and then that of the matching stage. Our analysis in the subsequent sections also follows this order.

1.1 Production stage: Technology

If two participants i, j with types $m, n \in \mathcal{M}$, respectively, are matched together, they engage in a partnership game⁶: they simultaneously choose effort levels $e_i, e_j \in \{E, S\}$ where E stands for exerting effort and S stands for shirking. Choice of E entails an “effort cost”

⁶This setup is a special case of a more general environment analyzed in Legros and Matsushima (1991). It is also related to the problem analyzed by Holmstrom (1982), though, as becomes clear below, it is different from Holmstrom (1982) in that effort choices are discrete and contracts are not restricted to be budget balanced.

of $c > 0$ while shirking is costless. For notational convenience, we summarize this via the function $C : \{E, S\} \rightarrow \mathbb{R}$, so that $C(E) = c$ and $C(S) = 0$. The output resulting from the effort choices is $y \in Y = \{\underline{y}, \bar{y}\}$ with $\bar{y} > \underline{y}$. The probability distribution over Y depends both on the types of the participants and the effort levels that they choose. This distribution is parameterized as follows:

$$(1) \quad \text{Prob}(\bar{y}|e_1, e_2, m, n) = \begin{cases} p(n, m) & \text{if } e_1 = e_2 = E \\ q_1(m, n) & \text{if } e_1 = E, e_2 = S \\ q_2(m, n) & \text{if } e_1 = S, e_2 = E \\ 0 & \text{if } e_1 = e_2 = S \end{cases}$$

We make the following assumptions:

A 1 For any $m, n \in \mathcal{M}$, $(p(m, n) - q_i(m, n))(\bar{y} - \underline{y}) > c$ and $q_i(m, n)(\bar{y} - \underline{y}) > c$, $i = 1, 2$.

A 2 $p(n, m)$ is strictly increasing in m and n , $q_i(m, n)$ are increasing in m and n , $i = 1, 2$.

A 3 For all $m, n \in \mathcal{M}$, $p(n, m) = p(m, n)$ and $q_1(m, n) = q_2(n, m)$.

A 4 For all $m, n \in \mathcal{M}$, $p(m, n), q_i(m, n) \in (0, 1)$.

Assumption **A1** implies that exerting effort is socially optimal for each agent in any match. Assumption **A2** says that an increase in the type of a partner raises the probability of high output for any combination of efforts: that is, higher types are more productive. Assumption **A3** requires that the roles of partner 1 and partner 2 are symmetric. In terms of productivity, only the type of a partner matters and not whether he is called partner 1 or partner 2. Assumption **A4** guarantees that deviations from effort to shirk are not perfectly detectable since both output levels have positive probability under profiles where at least one of the partners exerts effort. Finally, we assume that $p(\cdot, \cdot)$, and $q_i(\cdot, \cdot)$ are twice continuously differentiable.

1.2 Production stage: Contracts

Once an output is realized, it is shared according to a contract which the partners commit to. Effort choices are not publicly observable, and therefore are not contractible. Hence,

contracts can be written contingent only on the realized output. Correspondingly, a contract consists of effort level recommendations (e_1, e_2) for the two partners and a transfer scheme $(t_1, t_2) : Y \rightarrow \mathbb{R}^2$ specifying payments to each partner conditional on realized output. In what follows, we assume that enforceable transfers are those that satisfy a resource constraint: they can sum to no more than the total output produced in each contingency. That is, we consider transfers such that for all $y \in Y$, $t_1(y) + t_2(y) \leq y$. Notice that we allow for the total transfers to be strictly less than the realized output. In other words, we do not insist on balanced-budget contracts. If a contract involves $t_1(y) + t_2(y) < y$, we say that this contract involves “money burning”.

Assuming that partners 1 and 2 have types m and n , respectively, a contract (e_1, e_2, t) is incentive compatible if

$$(2) \quad e_1 \in \mathbf{argmax}_{e \in \{E, S\}} \sum_{y \in Y} \text{Prob}(y|e, e_2, m, n) t_1(y) - C(e)$$

and

$$(3) \quad e_2 \in \mathbf{argmax}_{e \in \{E, S\}} \sum_{y \in Y} \text{Prob}(y|e_1, e, m, n) t_2(y) - C(e)$$

This is the usual definition of incentive compatibility: under such a contract, given the opponent’s effort level and the types of both, each partner maximizes the expected transfers that he receives net of his effort cost, by choosing his part of the effort profile that is recommended. If a contract (e_1, e_2, t) is incentive compatible, we say that the transfer scheme t implements the effort profile (e_1, e_2) .

Finally, we say that a payoff vector (v_1, v_2) is feasible for a pair of agents with types m, n , respectively, if there exists an incentive compatible contract (e_1, e_2, t) such that:

$$v_1 = \sum_{y \in Y} \text{Prob}(y|e_1, e_2, m, n) t_1(y) - C(e_1)$$

and

$$v_2 = \sum_{y \in Y} \text{Prob}(y|e_1, e_2, m, n) t_2(y) - C(e_2)$$

Thus, for each combination of partner types (m, n) , the set of incentive compatible contracts generates the set of feasible payoff pairs. The properties of this set (namely, the dependence

of its Pareto frontier on the types of the partners) determine the outcome of the matching stage, which we describe next.

1.3 Matching stage: Partnership formation

The outcome of the market for partnerships is a matching defined as a one-to-one map $M : N \rightarrow N$ such that for any $i, j \in N$, $i = M(j)$ if and only if $j = M(i)$. The expression $M(i) = j$ is understood to mean that participants i and j form a partnership under matching M .

We are interested in the “stable” matchings of the partnership market in the standard sense:⁷ a matching is stable if no subsets of the participants can re-match among themselves so that each member of this subset improves their payoff.

Formally, an equilibrium consists of a matching $M : N \rightarrow N$ and a payoff assignment $v^* : N \rightarrow \mathbb{R}$ such that

- (1) for all i, j with $i = M(j)$, $(v^*(i), v^*(j))$ is feasible given their types;
- (2) the matching is stable: for any $i, j \in N$, there exists no payoff vector w which is feasible for i and j such that $w(i) > v^*(i)$ and $w(j) > v^*(j)$.

Our goal is to describe the properties of an equilibrium matching M . More specifically, we are interested in whether M exhibits positive assortative matching (when higher types are matched with higher types) or negative assortative matching (when higher types are matched with lower types).⁸ Clearly, which (if either) of these sorting patterns emerges as an equilibrium outcome depends on what sets of feasible payoffs are obtained under partnerships of various pairs of types. Note that part (2) of the definition of the equilibrium implies that any pair matched together in an equilibrium should receive a payoff pair that is on the Pareto frontier of payoffs that are feasible for them. Thus in the following section we derive the Pareto frontier for a given partnership.

⁷See for example Roth and Sotomayor (1990)

⁸Formally, M exhibits positive (respectively, negative) assortative matching if $m(i) > m(j)$ implies $m(M(i)) \geq m(M(j))$ (respectively, $m(M(i)) \leq m(M(j))$), where $m(i)$ refers to the type of participant i and therefore $m(M(i))$ is the type of participant i 's partner under matching M .

2 Characterization of the set of feasible payoffs

In this section we characterize the set of feasible payoff pairs for a partnership, in which partner 1 and partner 2 are of types $m \in \mathcal{M}$ and $n \in \mathcal{M}$ respectively. Without loss of generality, assume that $m \geq n$. The Pareto frontier of the set of feasible payoff pairs is found by solving the following problem:

$$\begin{aligned}
 (4) \quad W_{mn}(v) &= \max_{e_1, e_2, t_1(\cdot), t_2(\cdot)} \sum \text{Prob}(y|e_1, e_2, m, n) t_2(y) - C(e_2) \\
 &\text{s.t. } \sum \text{Prob}(y|e_1, e_2, m, n) t_1(y) - C(e_1) = v \\
 &e_1 \in \mathbf{argmax}_e \sum \text{Prob}(y|e, e_2, m, n) t_1(y) - C(e) \\
 &e_2 \in \mathbf{argmax}_e \sum \text{Prob}(y|e_1, e, m, n) t_2(y) - C(e) \\
 &t_1(y) + t_2(y) \leq y, \quad y = \underline{y}, \bar{y}
 \end{aligned}$$

The first constraint is the individual rationality constraint for partner 1, the second and the third are the incentive compatibility constraints and the last is the resource constraint. We solve (4) in two steps: first, for any effort profile (e_1, e_2) we find the optimal transfers implementing (e_1, e_2) ; then we determine which effort profile delivers the highest value to partner 2.

Before we describe the solution in detail, a few important observations can be made. Notice that, due to the possibility of making transfers, the set of payoff pairs obtained by contracts implementing a particular effort profile (e_1, e_2) has a linear frontier with the slope -1 .⁹ Denote the level of this frontier, which is the highest total surplus that can be achieved while implementing (e_1, e_2) , by $S^{e_1 e_2}(m, n)$. It is clear, then, that the solution to (4) has the form

$$(5) \quad W_{mn}(v) = -v + S(m, n),$$

⁹To see this, suppose that a pair of values (v, w) can be attained via an effort profile (e_1, e_2) and a transfer scheme $t_i(\bar{y}), t_i(\underline{y})$ ($i = 1, 2$) that implements (e_1, e_2) . Then, for any Δ , the combination of payoffs $(v + \Delta, w - \Delta)$ can be obtained via a contract consisting of the original effort profile (e_1, e_2) and transfers $(t_1(\bar{y}) + \Delta, t_1(\underline{y}) + \Delta)$ for agent 1 and $(t_2(\bar{y}) - \Delta, t_2(\underline{y}) - \Delta)$ for agent 2. Such a contract is incentive compatible since the effort levels and the spreads between individual transfers are the same as in the original contract. Moreover, this contract delivers the values $(v + \Delta, w - \Delta)$.

where $S(m, n) = \max\{S^{SS}(m, n), S^{EE}(m, n), S^{ES}(m, n), S^{SE}(m, n)\}$, and that all the payoff pairs on the Pareto frontier in (4) are achieved by contracts specifying the same effort profile. It follows from (5) that, in the terminology of the matching literature, our model has *fully transferable utility*, which substantially simplifies the characterization of the matching equilibrium in the next section. In order to provide full characterization of the Pareto frontier $W_{mn}(v)$ and to determine which effort combination is implemented along it, we find $S^{e_1 e_2}(m, n)$ for all $(e_1, e_2) \in \{E, S\}^2$.

If effort profile $(e_1, e_2) = EE$ is implemented, the optimal transfer scheme solves the following problem

$$\begin{aligned}
(6) \quad & \max_{t_1(\cdot), t_2(\cdot)} p(m, n)t_2(\bar{y}) + (1 - p(m, n))t_2(\underline{y}) - c \\
& \text{s.t. } p(m, n)t_1(\bar{y}) + (1 - p(m, n))t_1(\underline{y}) - c \geq v \\
& (p(m, n) - q_2(m, n))(t_1(\bar{y}) - t_1(\underline{y})) \geq c \\
& (p(m, n) - q_1(m, n))(t_2(\bar{y}) - t_2(\underline{y})) \geq c \\
& t_1(y) + t_2(y) \leq y, \quad y = \underline{y}, \bar{y}
\end{aligned}$$

Observe that if the partner 1's individual rationality constraint is slack, both transfers to the first agent may be decreased and the transfers to the second agent increased by equal amounts to make the individual rationality constraint binding. This adjustment will satisfy the incentive and resource constraints, and, at the same time, increase the payoff to the second agent. Therefore, the individual rationality constraint necessarily binds, and the optimal contract also maximizes the total surplus

$$S^{EE}(m, n) = p(m, n)[t_1(\bar{y}) + t_2(\bar{y})] + (1 - p(m, n))[t_1(\underline{y}) + t_2(\underline{y})] - 2c,$$

subject to the incentive compatibility and resource constraints.

Depending on the parameters of the model, the solution to (6) would either be budget balanced ($t_1(y) + t_2(y) = y$ for all y) or may involve money burning ($t_1(y) + t_2(y) < y$ for some y). If an incentive compatible budget contract exists, it delivers the surplus

$$(7) \quad S^*(m, n) = p(m, n)\bar{y} + (1 - p(m, n))\underline{y} - 2c.$$

Since $t_1(y) + t_2(y) \leq y$ for all y , this is the highest surplus that can potentially be achieved in the partnership. Note also that the same surplus would be obtained if effort levels were

observable. In other words, if budget balancing can be incentive compatible, there is no efficiency loss due to moral hazard. Since our goal is to analyze how inefficiencies resulting from moral hazard may affect sorting predictions, in the rest of the paper we restrict attention to the parameter values for which the solution to (6) requires money burning. Define

$$(8) \quad \eta(m, n) = \frac{c}{p(m, n) - q_1(m, n)} + \frac{c}{p(m, n) - q_2(m, n)} - (\bar{y} - \underline{y}).$$

The following assumption guarantees that the optimal contract involves money burning:

A 5 For any $m, n \in \mathcal{M}$, $\eta(m, n) > 0$.

Under assumption **A5**, any balanced budget transfer scheme would violate at least one of the incentive constraints, and thus the optimal contract must involve money burning.^{10,11} Correspondingly, when **A5** holds both incentive constraints bind. Also, to maximize the spread between the transfers in case of low and high output, money is burned only when low output is realized. This implies that $t_1(\bar{y}) + t_2(\bar{y}) = \bar{y}$, and therefore:

$$t_1(\underline{y}) + t_2(\underline{y}) = \underbrace{t_1(\bar{y}) + t_2(\bar{y})}_{\bar{y}} - \frac{c}{p(m, n) - q_2(m, n)} - \frac{c}{p(m, n) - q_1(m, n)} < \underline{y}.$$

The total surplus $S^{EE}(m, n)$ is then expressed as

$$(9) \quad \begin{aligned} S^{EE}(m, n) &= p(m, n)[t_1(\bar{y}) + t_2(\bar{y})] + (1 - p(m, n))[t_1(\underline{y}) + t_2(\underline{y})] - 2c \\ &= \underbrace{p(m, n)\bar{y} + (1 - p(m, n))\underline{y} - 2c}_{S^*(m, n)} - (1 - p(m, n))\eta(m, n) \end{aligned}$$

Notice that $S^{EE}(m, n)$ can be decomposed into two parts, the total surplus $S^*(m, n)$ that would be obtained if there were no moral hazard minus an additional term which appears due to moral hazard and is strictly positive (by assumption **A5**). This term represents the “expected moral hazard cost”: inefficient punishments of size $\eta(m, n) > 0$ occur when low output is realized, i.e. with frequency $(1 - p(m, n))$.

¹⁰This can easily be seen by adding up the incentive constraints, applying $t_1(y) + t_2(y) = y$ for all y and observing that the resulting inequality contradicts to **A5**.

¹¹The necessary and sufficient condition for inefficiency derived in Legros and Matsushima (1991) boils down to our assumption **A5** when applied to our setting. We formally demonstrate this in Section 7.1 of the Appendix.

The effort profile $(e_1, e_2) = ES$ can be implemented without money burning, and the partners obtain the total surplus of

$$(10) \quad \begin{aligned} S^{ES}(m, n) &= q_1(m, n) \underbrace{[t_1(\bar{y}) + t_2(\bar{y})]}_{\bar{y}} + (1 - q_1(m, n)) \underbrace{[t_1(\underline{y}) + t_2(\underline{y})]}_{\underline{y}} - c \\ &= q_1(m, n)\bar{y} + (1 - q_1(m, n))\underline{y} - c. \end{aligned}$$

Money burning is not required in this case because the incentive constraints of both partners can be satisfied by increasing the spread between the transfers to agent 1 and reducing the spread between the transfers to agent 2, while maintaining $t_1(y) + t_2(y) = y$ for all y .¹² Also note that $S^{ES}(m, n)$ can be rearranged as

$$(11) \quad S^{ES}(m, n) = S^*(m, n) - \underbrace{[(p(m, n) - q_1(m, n))(\bar{y} - \underline{y}) - c]}_{>0},$$

which, similarly to (9), is the difference between the surplus in the frictionless environment $S^*(m, n)$ and the cost of incentive provision, which, by assumption **A1**, is strictly positive. Notice, however, that the source of inefficiency in (11) is different from the one in (9): while under effort profile EE the moral hazard cost appears due to money burning, under ES its driving force is the inefficient effort choice.

By symmetry, under the effort profile $(e_1, e_2) = SE$, the partners can achieve the total surplus of

$$S^{SE}(m, n) = S^*(m, n) - [(p(m, n) - q_2(m, n))(\bar{y} - \underline{y}) - c].$$

Note also that, by assumption **A2**, $S^{ES}(m, n) \geq S^{SE}(m, n)$ if $m \geq n$.

Finally, if none of the partners exerts effort the total surplus is $S^{SS}(m, n) = \underline{y}$. Assumption **A1** ensures that $S^{ES}(m, n) > S^{SS}(m, n)$. Thus the total surplus obtained in the optimal contract is given by

$$(12) \quad \begin{aligned} S(m, n) &= \max\{S^{EE}(m, n), S^{ES}(m, n)\} \\ &= \underbrace{p(m, n)\bar{y} + (1 - p(m, n))\underline{y} - 2c}_{S^*(m, n)} - \underbrace{\min\{(1 - p(m, n))\eta(m, n), (p(m, n) - q_1(m, n))\bar{y} - c\}}_{\text{moral hazard cost}}. \end{aligned}$$

The following proposition summarizes the analysis of this section:

¹²In this case, the incentive constraints are $q_1(m, n)(t_1(\bar{y}) - t_1(\underline{y})) \geq c$ and $(p(m, n) - q_1(m, n))(t_2(\bar{y}) - t_2(\underline{y})) \leq c$ for agent 1 and agent 2, respectively.

Proposition 1 (Characterization of the set of feasible payoff pairs)

Suppose that assumptions **A1-A5** hold. Then for any $m, n \in \mathcal{M}$ such that $m \geq n$, the set of feasible payoffs for a partnership of types m and n is given by $\{(v, w) | v + w \leq S(m, n)\}$, where $S(m, n)$ is defined in (12). In addition, if

$$(13) \quad (1 - p(m, n))\eta(m, n) \leq (>) (p(m, n) - q_1(m, n))(\bar{y} - \underline{y}) - c,$$

where $\eta(m, n)$ is defined in (8), all the payoff pairs from the Pareto frontier of the set of feasible payoffs are obtained by the contracts implementing effort profile EE (ES).

Whether the optimal contracts induce effort by both partners or by only one of them, depends on the properties of the production technology. In subsequent sections, we alternately make one of the following two assumptions:

A 6 For all $m, n \in \mathcal{M}$ such that $m \geq n$:

$$(1 - p(m, n))\eta(m, n) \leq (p(m, n) - q_1(m, n))(\bar{y} - \underline{y}) - c$$

A 7 For all $m, n \in \mathcal{M}$ such that $m \geq n$:

$$(1 - p(m, n))\eta(m, n) > (p(m, n) - q_1(m, n))(\bar{y} - \underline{y}) - c$$

By Proposition 1, under assumption **A6**, effort profile EE is implemented in any partnership. Conversely, under assumption **A7**, effort profile ES is implemented.¹³ Note that assumption **A1** ensures that if $q_1(m, n)$ and $q_2(m, n)$ are sufficiently small for all $m, n \in \mathcal{M}$, then assumption **A6** holds.¹⁴ Intuitively, for such parameter values, incentives to exert effort are provided cheaply, because individual deviations lead to a large loss in total surplus and are not attractive, and thus effort profile EE is implemented in all partnerships. On

¹³Notice that we abstract from the cases when optimal effort profiles differ across partnerships. By focusing only on the two extreme situations, we are able to understand how particular distortions created by moral hazard—inefficient compensation or inefficient effort—may affect matching patterns. Potentially, interaction between the two sources of inefficiency may have additional effects on equilibrium matchings, but their analysis appears to be cumbersome and does not contribute to the main message of the paper, and are therefore omitted.

¹⁴As $q_i(m, n)$ become sufficiently close to 0, the inequality in assumption **A6** converges to $2c - p(m, n)c \leq p(m, n)(\bar{y} - \underline{y})$, which strictly holds by assumption **A1**.

the other hand, if $q_1(m, n)$ and $q_2(m, n)$ are sufficiently close to $p(m, n)$ for all $m, n \in \mathcal{M}$, assumption **A7** is satisfied. In this case, it becomes too expensive to induce effort by both partners and, at the same time, shutting one of them down leads to a relatively small loss in expected output, implying that all optimal contracts implement effort profile ES .

Alternatively, the values of $q_1(m, n)$ and $q_2(m, n)$ may be associated with the precision of the monitoring technology. When $q_1(m, n)$ and $q_2(m, n)$ are sufficiently small relative to $p(m, n)$, the likelihood ratios $L_i(m, n) = \frac{1-q_i(m, n)}{1-p(m, n)}$, $i = 1, 2$ are large, implying that the probability of punishment when it is warranted ($1 - q_i(\cdot, \cdot)$)—which is 1 minus the probability of committing a type II error when testing the hypothesis that one of the partners has deviated—is large relative to the probability of punishment when it is not warranted ($1 - p(\cdot, \cdot)$)—which is the probability of type I error when testing the hypothesis that one of the partners has deviated. In this case, the monitoring technology is of “*high precision*” and assumption **A6** is satisfied, while when the said likelihood ratio is small, the monitoring technology is of “*low precision*” and assumption **A7** is satisfied. In the next section, we take up these two cases (high precision and low precision monitoring) separately and describe the matching predictions under both types of technology.

3 Matching predictions

In this section we present our main results. In particular, we demonstrate that the presence of moral hazard has a non-trivial impact on sorting predictions and we provide economic intuition for it.

Proposition 1 establishes that, for any $m, n \in \mathcal{M}$, the sum of payoffs $S(m, n)$ along the Pareto frontier of the feasible payoff set is constant, i.e. this is a fully transferable utility environment. In this case, it is well-known (Becker (1973)) that for all equilibrium matchings to be positive (negative) assortative, it is sufficient that the function $S(\cdot, \cdot)$ is strictly supermodular (submodular). When the function $S(\cdot, \cdot)$ is twice continuously differentiable, an equivalent definition of supermodularity (due to Topkis (1978)) relates to the cross partial derivatives: $S(\cdot, \cdot)$ is strictly supermodular (submodular) if and only if $\frac{\partial^2 S(m, n)}{\partial m \partial n} > (<) 0$.¹⁵

¹⁵Strictly speaking, this characterization requires that $S(\cdot, \cdot)$ be twice continuously differentiable on $(a, b)^2$

Thus, our analysis in this section focuses on determining the sign of $\frac{\partial^2 S(m,n)}{\partial m \partial n}$.

In order to isolate the effects of moral hazard on sorting patterns, in the rest of the paper we assume that:¹⁶

$$\mathbf{A\ 8} \quad \frac{\partial^2 p(m,n)}{\partial m \partial n} = 0, \quad \frac{\partial^2 q_1(m,n)}{\partial m \partial n} = 0 \quad \text{and} \quad \frac{\partial^2 q_2(m,n)}{\partial m \partial n} = 0$$

This assumption guarantees that, in the absence of moral hazard, the total optimal surplus $S^*(m, n)$ defined in (7) would be neither submodular nor supermodular, and, correspondingly, any matching predictions that might arise in the model with imperfect monitoring are solely driven by unobservability of partners' efforts.

Next, we explore under which conditions $S(m, n)$ defined in Proposition 1 exhibits strict supermodularity or submodularity. We conduct the analysis separately for the two cases introduced in the previous section: *high precision monitoring* and *low precision monitoring*.

3.1 High precision monitoring

Throughout this subsection we maintain assumption **A6**. It follows from Proposition 1 that, in this case,

$$S(m, n) = \underbrace{p(m, n)\bar{y} + (1 - p(m, n))\underline{y}}_{S^*(m,n)} - 2c \dots$$

$$- \underbrace{(1 - p(m, n))}_{\text{frequency}} \left[\underbrace{\frac{c}{p(m, n) - q_2(m, n)} + \frac{c}{p(m, n) - q_1(m, n)} - (\bar{y} - \underline{y})}_{\eta(m,n), \text{ size of punishment}} \right]$$

As remarked above, under assumption **A8**, $S^*(m, n)$ exhibits no complementarities. Therefore, the possibility of super/sub modularity in $S(m, n)$ lies in the potential interaction of types of the two partners via the moral hazard cost, which is represented as the

 where (a, b) is an open interval in \mathbb{R} . In this case, the conclusion is that the function $S(\cdot, \cdot)$ is supermodular over $(a, b)^2$. Therefore, we are able to use this result, for instance, under the assumption that \mathcal{M} is an open interval.

¹⁶However, in the context of an example (Example 1) that violates this assumption (displaying technological complementarities between types) we discuss the interaction between technological complementarities and complementarities arising due to moral hazard.

product of the frequency and size of the inefficient punishment. In this subsection we take a closer look at this interaction.

First notice that by assumption **A2**, the frequency of inefficient punishment – which occurs only when the output is low – is decreasing in the types of both partners. The effect of a change in a partner’s type on the size of the punishment, however, is less straightforward. To see how this effect transpires, it is sufficient to analyze the term $\frac{c}{p(m,n)-q_2(m,n)}$. Notice that the denominator is related to the marginal product of effort for partner 1 when partner 2 is exerting effort: $(p(m,n) - q_2(m,n))(\bar{y} - \underline{y})$ measures the increase in expected output when partner 1 switches from shirking to working. If this term is increasing in type we say that *type and effort are complements*. If it is decreasing we say that *type and effort are substitutes*. Formally, in our characterization, we alternately make one of the following two assumptions:

A 9 (type and effort are complements) For all $m, n \in \mathcal{M}$,

$$\frac{\partial p(m, n)}{\partial m} - \frac{\partial q_2(m, n)}{\partial m} > 0 \text{ and } \frac{\partial p(m, n)}{\partial n} - \frac{\partial q_2(m, n)}{\partial n} \geq 0.$$

and

A 10 (type and effort are substitutes) For all $m, n \in \mathcal{M}$,

$$\frac{\partial p(m, n)}{\partial m} - \frac{\partial q_2(m, n)}{\partial m} < 0 \text{ and } \frac{\partial p(m, n)}{\partial n} - \frac{\partial q_2(m, n)}{\partial n} \leq 0.$$

Under assumption A9, effort is complementary to type: its marginal product is higher when the agent is of higher type or when matched with a higher type partner.¹⁷ In general, any technology of the form

$$(14) \quad \text{Prob}\{\bar{y}|e_1, e_2, m, n\} = m f_1(e_1, e_2) + n f_2(e_1, e_2) + p_0(e_1, e_2),$$

¹⁷Notice that the marginal product of effort of type m partner is given by $p(m, n) - q_2(m, n)$. We allow this marginal product to be constant in n , the type of the other partner; that is, marginal product of effort is allowed to be independent of the partner’s type. Also, notice that, by symmetry of the partners’ roles (assumption **A3**), it is implied that $\frac{\partial p(m, n)}{\partial n} - \frac{\partial q_1(m, n)}{\partial n} > 0$ and $\frac{\partial p(m, n)}{\partial m} - \frac{\partial q_1(m, n)}{\partial m} \geq 0$.

where $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ are increasing in both arguments, satisfies assumption **A9**. One canonical example that fits in this framework is where

$$p(m, n) = m + n + a_0 \quad q_1(m, n) = m + a_1 \quad q_2(m, n) = n + a_1,$$

with $a_i \in \mathbb{R}$, suitably chosen, i.e. the agent contributes his type to the production process only if he exerts effort. Another common example is

$$p(m, n) = m + n \quad q_1(m, n) = \alpha(m + n) \quad q_2(m, n) = \alpha(m + n),$$

where $\alpha \in (0, 1)$. In this case, both types contribute to the production process independently of whether the effort is exerted, but the outcome is augmented by the amount of total effort by the agents.

A more broad interpretation of Assumption **A9** is that ‘type’ plays the role of capital which the agent brings to the match, while ‘effort’ plays the role of not fully observable labor used (or cost needed) to operate this capital. For example, farmers who own different pieces of equipment used at various stages of farming (i.e. plowing, watering, mowing, etc.), can form a partnership to take advantage of using each other’s equipment. The output of such a partnership (harvest) is fully observed only in the end of the production process and often does not serve as a signal of strong / weak performance at any particular stage. The moral hazard problem arises because operating each piece of equipment is costly and the intensity of operation may be hard to monitor. In this case, farmer’s type (the quality of the equipment he owns) is obviously complementary to the effort he exerts (or the cost he incurs) while operating it. Another, perhaps more common, situation is one where the agent’s type is the amount of wealth he contributes to the partnership. The total partnership’s wealth is then used to purchase common equipment, which increases the productivity of individual effort inputs.¹⁸ The partnership’s success (e.g. sales of the final product or arrival of new clients) often cannot be attributed to the effort exerted by one or the other partner, thus creating a moral hazard problem. Examples of such partnerships can be found in software design, engineering, construction, etc. One more situation where observable type and unobservable

¹⁸The need to form a partnership may appear in the first place if there is a minimum start up cost that should be covered, and none of the partners has sufficient amount of wealth (owned or borrowed) that he is ready to allocate towards the business.

effort are complementary is common, is team sports, such as rowing or bobsleigh. Here the type of the agent is his athletic form, which clearly increases the productivity of his effort. The individual efforts cannot be monitored, and only the joint outcome (e.g. win or loss) is observable.

In contrast, if assumption **A10** holds, an agent's effort is a substitute for his and (weakly) his partner's type. The general form of technology that satisfies **A10** is

$$Prob\{\bar{y}|e_1, e_2, m, n\} = f_1(m + h_1(e_1, e_2)) + f_2(n + h_2(e_1, e_2)),$$

where $f_i(\cdot)$ and $h_i(\cdot, \cdot)$ are increasing, and $f_i(\cdot)$ are also concave. This could happen, for instance, if 'type' represents the agent's experience, while effort is exerted to acquire some project-specific information, and the probability of success is a decreasing returns to scale function of individual knowledge. Examples of such partnerships are quite common in consulting, legal or healthcare services. For instance, a group of business consultants work on a project proposal on restructuring / efficiency improvement for a potential client. Each consultant specializes in his part of the proposal (e.g. financial performance or human resources), and in the end the client either accepts or rejects an offer without specifying which part of the proposal was crucial for this decision. The more extensive the professional experience of each consultant (e.g. his knowledge of similar businesses), the less new information he might gather by studying the details of this particular business, thereby implying that experience and effort are substitutable. Clearly, individual effort levels are unobservable, leading to the type of the moral hazard problem studied in this paper. Similar situations occur when law partners cooperate on a case (since the final decision of the jury does not specify which part of the legal argument was the most persuasive), or when two healthcare professionals, such as a doctor and a physical therapist or a dietitian and a trainer, cooperate while treating a patient (since it is nearly impossible to determine which of the treatment procedures was responsible for the final outcome).

The following proposition summarizes our main characterization results. It establishes that the effect of moral hazard on equilibrium matching patterns depends on technological interaction between the agents' observable types and unobservable efforts (i.e. assumptions **A9** and **A10**).

Proposition 2 Assume **A1-A6**, and **A8** hold. Then,

1. Under **A9** all equilibrium matchings are negative assortative.

2. Under **A10**,

(a) all equilibrium matchings are positive assortative if $L_i(m, n) = \frac{1-q_i(m,n)}{1-p(m,n)}$ is increasing in m and n for $i = 1, 2$.

(b) all equilibrium matchings are negative assortative if $L_i(m, n) = \frac{1-q_i(m,n)}{1-p(m,n)}$ is decreasing in m and n for $i = 1, 2$.

Proof: We need to sign the expression $\frac{\partial^2 S(m,n)}{\partial m \partial n}$. Proposition 1 and assumption **A8** imply that

$$\frac{\partial^2 S(m, n)}{\partial m \partial n} = -\frac{\partial^2}{\partial m \partial n} \left[(1 - p(m, n)) \left(\frac{c}{p(m, n) - q_2(m, n)} + \frac{c}{p(m, n) - q_1(m, n)} \right) \right].$$

Then, by symmetry, the sign of $\frac{\partial^2 S(m,n)}{\partial m \partial n}$ is the negative of the sign of $\frac{\partial^2}{\partial m \partial n} (1-p(m, n)) \frac{c}{p(m,n)-q_2(m,n)}$.

For brevity, we drop reference to m and n , and express¹⁹

$$(15) \quad \frac{\partial^2}{\partial m \partial n} \frac{1-p}{p-q_2} = - \left[\frac{\partial}{\partial m} (p-q_2) \cdot \frac{\partial}{\partial n} \frac{1-p}{p-q_2} + \frac{\partial}{\partial n} (p-q_2) \cdot \frac{\partial}{\partial m} \frac{1-p}{p-q_2} \right] / (p-q_2).$$

For part 1, notice that assumption **A9** implies $\frac{\partial}{\partial m} (p-q_2) > 0$ and $\frac{\partial}{\partial n} (p-q_2) \geq 0$. In addition, since $1-p(m, n)$ is decreasing in m and n , it also follows that $\frac{\partial}{\partial n} \frac{1-p}{p-q_2} < 0$ and $\frac{\partial}{\partial m} \frac{1-p}{p-q_2} < 0$. Thus $\frac{\partial^2}{\partial m \partial n} \frac{1-p}{p-q_2} > 0$, implying negative assortative matching.

For part 2, note that by assumption **A10**, $\frac{\partial}{\partial m} (p-q_2) < 0$ and $\frac{\partial}{\partial n} (p-q_2) \leq 0$. Thus, if $\frac{1-p}{p-q_2}$ is strictly decreasing in m and n , then $\frac{\partial^2}{\partial m \partial n} \frac{1-p}{p-q_2} < 0$. Noting that $L_i(m, n) = \frac{1-q_i(m,n)}{1-p(m,n)}$ is inversely related to $\frac{1-p(m,n)}{p(m,n)-q_i(m,n)} = \frac{1}{L_i(m,n)-1}$ establishes part 2(a). The argument for part 2(b) is analogous. \square

The intuition behind Proposition 2 is as follows: An equilibrium matching maximizes the total surplus. Hence, it minimizes the sum of the costs of incentive provision ($1 -$

¹⁹To verify this, observe that for any $f(m, n)$ and $g(m, n)$ such that $f_{mn}(m, n) = 0$ and $g_{mn}(m, n) = 0$ for all m, n , the following holds: $\frac{\partial}{\partial m} \frac{f}{g} = \frac{f_m - g_m \frac{f}{g}}{g}$, and thus $\frac{\partial^2}{\partial m \partial n} \frac{f}{g} = - \left[g_m g \frac{\partial}{\partial n} \frac{f}{g} + g_n \left(f_m - g_m \frac{f}{g} \right) \right] / g^2$, which simplifies to $\frac{\partial^2}{\partial m \partial n} \frac{f}{g} = - \left[g_m \frac{\partial}{\partial n} \frac{f}{g} + g_n \frac{\partial}{\partial m} \frac{f}{g} \right] / g$.

$p(n, m))\eta(m, n)$ across matched pairs. On the one hand, the frequency of inefficient punishments $1 - p(n, m)$ is decreasing in the types of both partners. On the other hand, the size of the punishment $\eta(m, n)$ is inversely related to $p(m, n) - q_2(m, n)$, which may be increasing or decreasing in types m and n , depending on whether **A9** or **A10** holds. It is instructive to analyze the following identity, which holds under assumption **A8**, to understand the interaction between these forces:

$$\frac{\partial^2}{\partial m \partial n} \left[(1 - p) \times \frac{c}{p - q_2} \right] =$$

$$(16) \quad \frac{\partial}{\partial n} (1 - p) \times \frac{\partial}{\partial m} \frac{c}{p - q_2} + \frac{\partial}{\partial m} (1 - p) \times \frac{\partial}{\partial n} \frac{c}{p - q_2} + (1 - p) \times \frac{\partial^2}{\partial m \partial n} \frac{c}{p - q_2}$$

This identity decomposes the cross-partial of $\left[(1 - p) \times \frac{c}{p - q_2} \right]$ into three additive terms: first and second representing the interaction between the impact of the type of one partner on the size of punishment and the other on the frequency of punishment, while the third term represents the interaction of the types of the two sides through their impact on the size of the punishment.

Since the frequency of punishment $(1 - p(m, n))$ is decreasing in both m and n , the first two of these additive terms would be positive and therefore push towards positive assortative matching when the effort and type are complements (i.e. when $p(m, n) - q(m, n)$ is increasing in m and n as under assumption **A9**).

Intuitively, the fact that $p(m, n) - q_2(m, n)$ is increasing in m implies that higher types have less incentive to deviate (because their deviations have larger effects on the total expected output), and thus the optimal contract punishes them less in the event of low output. By the same token, that this term is (weakly) increasing in n implies that the partners of the higher types are faced with lower punishments. To minimize the cost of incentive provision, lower types, who lead to more frequent punishments, should be matched with higher types, who induce smaller size punishment. Thus the interaction between the frequency and the size of punishment acts in favor of negative matching.

When assumption **A10** replaces assumption **A9**, higher types, as well as their partners, lead to bigger-size punishments. Thus, the intuition is reversed and to minimize the cost

of incentive provision, higher types should be matched with higher types so that bigger size punishments occur less frequently. This, in this case, acts in favor of positive assortative matching.

The third term in (16), on the other hand, always pushes toward negative assortative matching. To see this note that this term can further be expressed as

$$\frac{\partial^2}{\partial m \partial n} \frac{c}{p - q_2} = \frac{1}{(p - q_2)^3} \times \frac{\partial}{\partial m} (p - q_2) \times \frac{\partial}{\partial n} (p - q_2) > 0,$$

which is positive as long as the derivatives of $p(m, n) - q(m, n)$ with respect to m and n have the same sign.

Intuitively, notice that the size of the punishment $\frac{1}{p(m, n) - q(m, n)}$ is a convex decreasing function of the marginal product of effort $p(m, n) - q(m, n)$. Consider for instance the case where this quantity is increasing in m and n (assumption **A9**). Even though the size of the punishment is decreasing in the type of both partners, the marginal impact on this size of increasing the type of, say, partner 1 is smaller when $p(m, n) - q(m, n)$ is already large; i.e. the punishment is already small. This is due to the convexity of the size in $p(m, n) - q(m, n)$. This implies that, in terms of minimizing the total expected punishment it is more efficient to match higher types with lower types. A similar intuition applies under assumption **A10**.

Now, clearly, under assumption **A9**, all three terms are positive and therefore, negative assortative matching unambiguously obtains. Under assumption **A10**, on the other hand, the race between the first two terms versus the third term determines the equilibrium matching patterns. To see why the behavior of the likelihood ratio $L_i(m, n) = \frac{1 - q_i(m, n)}{1 - p(m, n)}$ plays a role in determining the winner in this race, notice once again that the interaction between the types through the frequency and the size of the punishment pushes towards positive assortative matching. Intuitively, this impact will be larger when the variation in the frequencies of inefficient punishments across types (for a given variation in sizes) is large. Observe that $L_i(m, n)$ may be increasing only if $1 - p(m, n)$ declines with types sufficiently fast, that is, only if the probability of inefficient punishments varies a lot across different types.²⁰ Thus increasing likelihood ratio is associated with positive assortative matching. In contrast, for

²⁰This is because, the likelihood ratio $L_i(m, n) = \frac{1 - q_i(m, n)}{1 - p(m, n)}$ is inversely related to $\frac{1 - p(m, n)}{p(m, n) - q_i(m, n)}$, where both the numerator and the denominator are decreasing in m and n .

$L_i(m, n)$ to be decreasing, it is necessary that $1 - p(m, n)$ declines with types sufficiently slowly, which leads to relatively little variation in the frequency of punishment across types. Thus the impact of the interaction of types through frequency and size is strong. This is why decreasing likelihood ratio is associated with negative assortative matching.

Finally, an alternative interpretation of the mechanism through which the interaction between the frequency and the size of inefficient punishments generates matching predictions is in terms of the willingness and ability of each type agent to transfer additional payoffs to his partner to be able to switch to a higher type one. Here, we discuss the case where **A9** holds. The alternative case with **A10** holding is analogous. Notice that in a stable matching, those who are willing and able to pay more for such a switch end up with higher type partners. Switching to a higher type partner is good for all types because, by assumption **A9**, it reduces the size of inefficient punishments. However, this reduction in the size of the punishment is more important for the lower types since they have to go to these punishments more often. Moreover, such a switch will decrease the frequency of punishment, which is, once again, more important for a lower type agent since the size of the total inefficient punishment is decreasing in type (assumption **A9**). This is why lower types would be willing to ‘bid more’ for a higher type partner, which leads to negative assortative matching in equilibrium. To be more concrete, consider a simple technology where $p(m, n) = m + n$, $q_1(m, n) = m$ and $q_2(m, n) = n$. In this case, the total size of the inefficient punishment for a matched pair of types m, n is $\frac{c}{m} + \frac{c}{n} - (\bar{y} - y)$. Take two types $n > n'$. If any participant—independent of his type—switches from an n' -type partner to an n -type partner, the total inefficient punishment in the partnership will fall by $\frac{c}{n'} - \frac{c}{n}$ and the frequency of inefficient punishment will fall by $n - n'$. Therefore, to make such a switch, an agent of type m would be willing to leave a value of $(1 - m) \left(\frac{c}{n'} - \frac{c}{n} \right) + (n - n') \frac{c}{m} + u_{n'}$ to the n -type agent, where $u_{n'}$ is the payoff of a type n' agent. Notice that this is decreasing in m : exactly because the reduction in size as well as the reduction in frequency is more important for a lower type agent.

Remark 1 Our results do not require that an agent’s marginal product of effort depends on his partner’s type. Observe that if the marginal product of the agent’s effort depends on his own type but not on the type of his partner (i.e. $\frac{\partial p(m, n)}{\partial n} - \frac{\partial q_2(m, n)}{\partial n} = 0$), the last two terms in

(16) disappear, but the first term remains strictly positive under assumption **A9** and strictly negative under assumption **A10**, implying that $S(m, n)$ is strictly submodular in the former case and supermodular in the latter. In other words, the non-trivial matching predictions persist even if there is no interaction between an agent's effort and his partner's type. In fact, under assumption **A10** the results become stronger: when $\frac{\partial p(m, n)}{\partial n} - \frac{\partial q_2(m, n)}{\partial n} = 0$, **A10** unambiguously leads to negative sorting in equilibrium.

To sum up, Proposition 2 illustrates that, under sufficiently precise monitoring—i.e. when assumption **A6** holds—the effects of moral hazard on matching predictions depend on the properties of the underlying technology. When unobservable effort and observable type are complements, moral hazard leads to negative assortative matching, but when effort and type are substitutes, moral hazard may lead to positive assortative matching. Recall that these results are derived under the assumptions of no complementarity in underlying technology (assumption **A8**). Had such complementarity been present (i.e. if $\frac{\partial^2 p(m, n)}{\partial m \partial n} \neq 0$), moral hazard could either reinforce it or act against it, depending on the technological properties.²¹ In fact, in the following example we illustrate a situation where introduction of moral hazard reverses Becker's prediction as long as the complementarity embodied in the production technology is not too strong, in a sense that we make precise below.

Example 1 Consider an environment where $q(m, n) = 0$ for all m, n . That is, no success can be achieved unless both agents exert effort. This can be interpreted as an extreme form complementarity between the effort choices of the partners. In this case, the marginal product of effort when the partner is exerting effort is simply given by $p(m, n)$, which is increasing in both m and n . Therefore, our model predicts negative assortative matching in this environment whenever $\frac{\partial^2}{\partial m \partial n} p(m, n) = 0$. In this example, we allow $\frac{\partial^2}{\partial m \partial n} p(m, n) > 0$ which, in the absence of moral hazard, would imply positive assortative matching in all equilibria.

²¹Namely, the expression for $\frac{\partial^2 S(m, n)}{\partial m \partial n}$ would have an additional non-zero term $\frac{\partial^2 p(m, n)}{\partial m \partial n} \times \left(\frac{c}{p(m, n) - q_1(m, n)} + \frac{c}{p(m, n) - q_2(m, n)} - (\bar{y} - \underline{y}) \right)$ in (16), which has the same sign as $\frac{\partial^2 p(m, n)}{\partial m \partial n}$.

In this context,

$$\begin{aligned} S(m, n) &= p(m, n)\bar{y} + (1 - p(m, n))\underline{y} - 2c - (1 - p(m, n)) \left[\frac{c}{p(m, n)} + \frac{c}{p(m, n)} - (\bar{y} - \underline{y}) \right] \\ &= \bar{y} - \underline{y} - 2c - 2c \frac{1 - p(m, n)}{p(m, n)} \end{aligned}$$

Correspondingly,

$$\frac{\partial^2}{\partial m \partial n} S(m, n) = \frac{2c}{p(m, n)^3} \left[p(m, n) \frac{\partial^2 p(m, n)}{\partial m \partial n} - 2 \frac{\partial p(m, n)}{\partial m} \frac{\partial p(m, n)}{\partial n} \right]$$

Therefore, all equilibrium matchings will be negative assortative as long as

$$\left[p(m, n) \frac{\partial^2 p(m, n)}{\partial m \partial n} \right] / \left[\frac{\partial p(m, n)}{\partial m} \frac{\partial p(m, n)}{\partial n} \right] \leq 2$$

Notice that this restriction allows not only supermodularity of $p(m, n)$ but also a stronger form of supermodularity, namely, log-supermodularity of $p(m, n)$ while still predicting negative assortative matching as the unique equilibrium outcome.²² Therefore, this is an environment where even log-supermodularity of the underlying technology is not sufficient for positive assortative matching to obtain in equilibrium in the presence of moral hazard. This is a strong reversal of Becker's prediction.²³

3.2 Low precision monitoring

Now we turn to the alternative assumption **A7**. As shown in Section 3, under this restriction

$$S(m, n) = q_1(m, n)\bar{y} + (1 - q_1(m, n))\underline{y} - c$$

Recall that this surplus is obtained via a contract that prescribes effort only by the higher type partner, and the only loss of efficiency is due to inefficient effort—no inefficient punishments are necessary. Moreover, in this case, in an equilibrium, exactly half of the population exerts effort in the production stage, while the other half remains idle. Given this, the total surplus is maximized under matchings which have the following property: if agents are ranked according to their types, each agent that is ranked above the median

²²Recall that log-supermodularity of $p(m, n)$ is equivalent to $\left[p(m, n) \frac{\partial^2 p(m, n)}{\partial m \partial n} \right] / \left[\frac{\partial p(m, n)}{\partial m} \frac{\partial p(m, n)}{\partial n} \right] \geq 1$.

²³Log-supermodularity is a strong form of complementarity and is known to restore positive assortative matching in environments with search frictions. See Shimer and Smith (2000).

will be matched with an agent that is ranked below the median. This type of matching will guarantee that the most productive half of the population exerts effort and the least productive half remains idle. Therefore, any stable matching will exhibit this property, since stable matchings maximize the total surplus across matchings. That is, all equilibrium matchings will feature “across the median matches”.²⁴

This type of sorting predictions are obtained in a recent paper by Citanna and Chakraborty (2005) in a model where the roles of the partners are exogenously asymmetric. In their model, the agents are heterogenous with respect to their wealth levels and they write contracts that respect limited liability to provide incentives for effort. Since the limited liability constraint is less stringent for richer agents, it is more efficient to allocate them to the more effort-intensive roles where it is necessary to provide larger wedges between payments in case of success and in case of failure. In some sense, when assumption **A7** holds in our model, the partners’ roles become endogenously asymmetric, and the matching predictions resemble those in Citanna and Chakraborty (2005).

4 Money burning versus dynamic incentives

The assumption that two partners that are matched together can commit to discard some of the output may appear unrealistic. Even though this is a necessary assumption to obtain our results when the matched partners are assumed to engage in a one-shot productive relationship, it is possible to dispense with it if the partners are assumed to play an infinitely “repeated partnership game” once they are matched.

In the Appendix (Section 7.2) we formally develop such a model of a repeated partnership game in which the technology of production is identical to the one in our main model described in Section 1. Once two agents are matched together, they play a repeated game

²⁴Notice that this type of matching has a flavor of “negative sorting”. In particular, it would be equivalent to negative assortative matching if there were only two types of agents. When there are more types, it rules out positive assortative matching. For instance, if there are 4 agents with types $m_1 > m_2 > m_3 > m_4$, all possible matchings except the positive assortative matching (where m_1 matches m_2 and m_3 matches m_4) satisfy this condition. In particular, a matching where m_1 matches m_3 and m_2 matches m_4 (as well as the negative assortative matching where m_1 matches m_4) would be an “across the median” matching.

in every stage of which they simultaneously choose effort levels. After the realization of output, they make simultaneous transfers to each other but they cannot discard resources. We also assume that the partners cannot commit to contracts: the transfers that they make in equilibrium need to be self-enforcing. The history of outputs and transfers are publicly observable, while the effort levels are not. Therefore, this is a repeated game with imperfect monitoring. Finally, we assume that the partners discount their future payoffs using a common discount factor δ . We take the set of feasible payoff vectors for a matched pair to be the set of “public perfect equilibrium” (PPE) payoff vectors of this game.²⁵

In this context, we establish the following result, which is an analog of Proposition 2 in the main text:

Proposition 3 (*Matching predictions in a repeated partnership game under precise monitoring*) Assume **A1-A5**, **A6** and **A8** hold.

- (a) *If, additionally, **A9** holds, then there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, all equilibrium matchings are negative assortative.*
- (b) *If additionally, **A10** holds, then there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, all equilibrium matchings are positive (negative) assortative if $L_i(m, n) = \frac{1-q_i(m,n)}{1-p(m,n)}$ is increasing (decreasing) in m and n , $i = 1, 2$.*

The main difficulty with establishing the results of Proposition 3 is due to the fact that we do not have the exact characterization of the Pareto frontiers for δ strictly smaller than 1, even if it is arbitrarily close to 1. In particular, we know that these frontiers are not linear.²⁶ Therefore, for any $\delta < 1$, the environment is not one of transferable utility, and

²⁵PPE are equilibria in which all strategies are conditioned solely on publicly observable occurrences – in this case, the history of observed outputs. See Appendix for the formal definition.

²⁶It can be shown that for each (large enough) δ , there is a piece of the frontier which is linear with slope -1, but close to the axes the frontier becomes strictly concave. The reason, roughly, is as follows: when the value of one of the partners is very low (lower than $(1 - \delta)\frac{\bar{v}}{2}$) it is not possible to give him incentives to make the necessary transfers to achieve this value using trigger strategies: he is better off not making the transfer in case of success, keeping his half of the output and getting 0 payoff from then on. Therefore, to deliver such a value, the partners need to use suboptimal effort choices. That is why, the level of the frontier ($v + W(v)$) is lower at points close to the axes. As $\delta \rightarrow 1$, the “concave” part of the frontier disappears and

hence the technique of using the link between increasing/decreasing differences and matching patterns is not valid. Instead, we use a generalized version of this link, established by Legros and Newman (2007), to show our result. In Lemma 2 in the Appendix we show that if the Pareto frontier of the limit set (as $\delta \rightarrow 1$) of equilibrium payoff vectors has transferable utility and satisfies increasing/decreasing differences, then, for all sufficiently large δ , the Pareto frontiers of the actual sets of equilibrium payoffs satisfy *generalized* increasing/decreasing difference, which allows us to apply the results in Legros and Newman (2007) and conclude that the equilibrium matchings are positive/negative assortative.

Then, in order to characterize the properties of the limit Pareto frontier, we employ the methodology developed by Fudenberg et al. (1994). Proposition 5 in the Appendix establishes that the limit Pareto frontier coincides with the Pareto frontier of feasible payoffs in the static partnership game that we analyze in the main text (see Proposition 1). This result is intuitive: in this setup, money burning is replaced by moving to an equilibrium whose payoffs are off-the Pareto frontier and commitment to contracts is replaced by implementation via trigger strategies (since transfers are observable). Since we know that the Pareto frontier obtained in the static model has transferable utility and satisfies increasing (decreasing) differences when assumption **A9** (**A10**) holds, the argument in the above paragraph implies that the first (second) statement of Proposition 3 applies. The formal analysis supporting the above discussion is included in the Appendix (Sections 7.2.3-7.2.4).

5 Matching under an alternative organizational structure

The main goal of this section is to illustrate that matching predictions that arise due to moral hazard vary across different organizational structures. In particular, instead of looking at the formation of partnerships, we analyze the decision of a firm owner organizing his employees into teams and show that under some restrictions on the parameter values, the impact of moral hazard on matching patterns in these two organizational forms are exactly opposite.

the limit frontier becomes linear with slope -1.

To draw this contrast, we compare our results to those obtained in a modified version of the model in recent work by Franco et al. (2008). It is important to point out that our goal here is not to explain why a firm may originate, but rather to describe how a firm owner sorts its workers in teams, provided that the firm already exists. At the same time, as we discuss later, our results in this section may also have interesting implications for endogenous appearance of firms versus partnerships in different technological environments, implying that, while endogenizing the organizational structure is outside the scope of this paper, it may be an interesting direction for future research.

Suppose, as in Franco et al. (2008), that instead of participating in the market for partnerships, all the agents work in one firm, and the firm owner (the principal) organizes them in teams and offers a wage profile contingent on all observable variables. Assume that workers' outside opportunity is given by $\underline{u} \geq 0$ and that they have limited liability, in the sense that wages cannot be below zero in any state.

Then the profit $\Pi(m, n)$ extracted by the firm owner from a team of workers with types m and n is found by solving the following problem:

$$\begin{aligned}
 \Pi(m, n) = & \max_{e_1, e_2, w_1(\cdot), w_2(\cdot)} \sum \text{Prob}(y|e_1, e_2, m, n)(y - w_1(y) - w_2(y)) \\
 \text{s.t. } & e_1 \in \mathbf{argmax}_e \sum \text{Prob}(y|e, e_2, m, n)w_1(y) - C(e) \\
 & e_2 \in \mathbf{argmax}_e \sum \text{Prob}(y|e_1, e, m, n)w_2(y) - C(e) \\
 & \sum \text{Prob}(y|e_1, e_2, m, n)w_i(y) - C(e_i) \geq \underline{u}, \quad i = 1, 2 \\
 & w_i(y) \geq 0 \text{ for all } y, \quad i = 1, 2
 \end{aligned}
 \tag{17}$$

where $w_1(y)$ and $w_2(y)$ are output-contingent wages of agent 1 and agent 2 respectively. The principal collects the output net of the total wage payment. The first two constraints are the incentive compatibility constraints for the workers, while the last two constraints are the individual rationality and limited liability constraints.²⁷

As in the solution for the partnership's optimal contract, we find the optimal compen-

²⁷This is a modification of the model studied in Franco et al. (2008). They analyze the model with continuous effort choice, where productivity differences arise endogenously due to differences in effort costs. In our case, discreteness of the effort allows for a clearer comparison of the two environments, while still capturing the main mechanism leading to contrasting matching predictions.

sation scheme implementing any given effort profile, and then choose the effort profile that delivers the highest profit. To illustrate our point, we restrict attention to cases where the following assumption holds for all $m, n \in \mathcal{M}$:

$$(18) \quad p(m, n) \frac{c}{p(m, n) - q_1(m, n)} - c \geq \underline{u} \quad \text{and} \quad p(m, n) \frac{c}{p(m, n) - q_2(m, n)} - c \geq \underline{u}$$

As we show below, this assumption guarantees that for the contract implementing the effort profile (E, E) , the individual rationality constraints do not bind (because the workers' outside value is sufficiently low).²⁸

Let $\Pi^{e_1, e_2}(m, n)$ stand for the principal's maximum profit generated by a team of types (m, n) , conditional on implementing effort profile (e_1, e_2) . If none of the workers is asked to exert effort then $\Pi^{SS}(m, n) = \underline{y} - 2\underline{u}$. If only the first agent exerts effort, the principal can capture the full surplus net of the agents' outside options by setting $w_1(\underline{y}) = \underline{u}$, $w_1(\bar{y}) = \frac{c}{q_1(m, n)} + \underline{u}$ and $w_2(y) = \underline{u}$ for all y . Thus

$$\begin{aligned} \Pi^{ES}(m, n) &= q_1(m, n) \left(\bar{y} - \frac{c}{q_1(m, n)} \right) + (1 - q_1(m, n))\underline{y} - 2\underline{u} \\ &= q_1(m, n)\bar{y} + (1 - q_1(m, n))\underline{y} - c - 2\underline{u} \\ &> \underline{y} - 2\underline{u} = \Pi^{SS}(m, n), \end{aligned}$$

where the last inequality is a direct implication of assumption **A1**. Similarly, if only the second agent exerts effort, $\Pi^{SE}(m, n) = q_2(m, n)\bar{y} - c - 2\underline{u}$, which, by assumption **A2**, is smaller than $\Pi^{ES}(m, n)$ if $m \geq n$.

If both agents exert effort, in the optimal contract either the limited liability constraint or the participation constraint of the agents bind. It is easy to see that the optimal contract, ignoring the participation constraint, is where the principal sets $w_i(\underline{y}) = 0$ and $w_i(\bar{y}) = \frac{c}{p(m, n) - q_i(m, n)}$, $i = 1, 2$, and hence both incentive compatibility constraints bind. By the assumption stated in (18), the participation constraint is also satisfied by this contract. Therefore, the maximal profit that the principal earns while implementing effort profile EE

²⁸If the individual rationality constraints were binding for all the workers in all matches, the profit of the firm owner would be independent of the sorting structure, implying that moral hazard would have no impact on matching predictions.

is given by:

$$(19) \quad \Pi^{EE}(m, n) = p(m, n) \left(\bar{y} - \frac{c}{p(m, n) - q_1(m, n)} - \frac{c}{p(m, n) - q_2(m, n)} \right) + (1 - p(m, n))\underline{y}$$

In the rest of this section, we focus on the set of parameters, for which $\Pi^{EE}(m, n) \geq \Pi^{ES}(m, n)$ for all $m \geq n$, because the most interesting comparisons between the two environments are drawn in this case.²⁹ It is easy to verify that this occurs if the following inequality holds for all $m, n \in \mathcal{M}$ such that $m \geq n$:

$$(20) \quad 2\underline{u} \geq p(m, n) \left(\frac{c}{p(m, n) - q_1(m, n)} + \frac{c}{p(m, n) - q_2(m, n)} \right) - c - (p(m, n) - q_1(m, n))(\bar{y} - \underline{y}),$$

i.e. if the monitoring technology is sufficiently precise.³⁰

The optimal matching structure maximizes the total profit of the firm owner $\sum_{\text{all matches}} \Pi(m, n)$, or minimizes the cost of total payouts (wages paid when high output is realized), which under condition (20) and assumption **A8** simplifies to

$$\sum_{\text{all matches}} p(m, n) \left(\frac{c}{p(m, n) - q_2(m, n)} + \frac{c}{p(m, n) - q_1(m, n)} \right).$$

Recall that equilibrium matchings in the partnership model minimize the total cost associated with inefficient punishments, which, in case of precise monitoring, is given by

$$\sum_{\text{all matches}} (1 - p(m, n)) \left(\frac{c}{p(m, n) - q_2(m, n)} + \frac{c}{p(m, n) - q_1(m, n)} - (\bar{y} - \underline{y}) \right).$$

Observe that, the payouts in the firm owner's problem, as well as the inefficient punishments in the partnership model, are determined by the wedge between payments to a given type worker, which are necessary to bind his incentive constraint. Therefore, for a given pair of workers who are matched together, the necessary payouts in the firm owner's problem and the necessary punishment in the partnership model are the same up to an additive constant. Now, while the payouts in the firm owner's problem are made when *high* output is realized

²⁹If effort profile *ES* is chosen for all pairs in both organizational structures, the principal's profit $\Pi(m, n)$ coincides with the surplus $S(m, n)$ obtained in the partnership up to an additive constant $2\underline{u}$, and thus the matching structures in the two models have the same properties.

³⁰It is easy to see that condition (20) is compatible with (18) since $(p(m, n) - q_1(m, n))(\bar{y} - \underline{y}) > c$, by assumption A1.

(i.e. with probability $p(m, n)$), the inefficient punishments in the partnership model happen when *low* output occurs (i.e. with probability $1 - p(m, n)$). This distinction has important implication for matching predictions. Namely, in order to maximize profit, the principal would like to team the workers who generate larger size payouts with the workers who lead to smaller probability of these payouts: that is *lower* type workers. Similarly, in equilibria of the partnership model the workers who generate larger size punishments are matched with workers that lead to smaller probability of these punishments, who, in sharp contrast to the firm's problem, are the *higher* types. This implies that opposite matching patterns may arise in the two models. This, nevertheless, will not always be so because, in addition to the interaction between the frequency and the size of payouts, there is also the effect of both workers' types on the size of the payout (described by $\frac{\partial^2}{\partial m \partial n} \frac{c}{p(m, n) - q_i(m, n)}$, $i = 1, 2$), which, as in the model of partnerships, always favors negative sorting. However, if this effect is relatively weak (or disappears, as in the case when the marginal product of effort depends only on the agent's own type – see Remark 1), the two organizational structures generate the opposite matching predictions. The following proposition formally summarizes how matching patterns chosen by the firm owner vary with the properties of technology.

Proposition 4 (*Matching patterns within a firm*)

Suppose that **A1** - **A6** and **A8** hold and that conditions (18) and (20) is satisfied for all $m \geq n$.

(a) If, additionally, assumption **A9** holds, i.e. effort and type are complements, then the optimal matching of workers' types within the firm exhibits positive (negative) assortative matching if $\frac{p(m, n)}{p(m, n) - q_i(m, n)}$ is increasing (decreasing) for all $m, n \in \mathcal{M}$.³¹

(b) If, additionally, assumption **A10** holds, i.e. effort and type are substitutes, then the

³¹In contrast to Proposition 2, we cannot formulate part (a) of Proposition 4 in terms of the likelihood ratio, since monotonicity of $L_i(m, n)$ is not equivalent to the monotonicity of $\frac{p(m, n)}{p(m, n) - q_i(m, n)}$ (and it is the latter that stipulates the results). Nevertheless, the intuition for the results in (a) relies on the same mechanisms as outlined after Proposition 2: under assumption **A9**, increasing $\frac{p(m, n)}{p(m, n) - q_i(m, n)}$ implies that the frequency of payouts varies a lot across agents of different types, and thus the interaction between the frequency and the size of payouts becomes the driving force of sorting of types in teams; in contrast, when $\frac{p(m, n)}{p(m, n) - q_i(m, n)}$ is decreasing, there is not much variation in frequency, and thus the matching patterns are driven by the interaction between the sizes of payouts.

optimal matching of workers' types within the firm exhibits negative assortative matching.

The sharp contrast between the matching predictions arising within the firms and in the market for partnerships has a potentially interesting application regarding the appearance of different organizational structures in different technological environments. Consider, for example, the case when effort and type are complements (i.e. assumption **A9** holds). Suppose also that the underlying technology has some degree of complementarity (i.e. $\frac{\partial^2 p(m,n)}{\partial m \partial n} > 0$), which is perhaps a natural feature in the team production models. Under assumption **A9**, firms benefit from complementarity in $p(m,n)$ more than partnerships do. The reason is that, from the perspective of the firm owner, positive matching simultaneously maximizes the gains from complementarity in the underlying technology and minimizes the losses due to unobservable effort. The partnership market, however, faces a trade-off between these two channels: on the one hand, the gains from embodied technological complementarities are maximized via positive matchings, but, on the other hand, the interaction between the frequency and the size of inefficient punishment due to moral hazard is fully exploited via negative matchings. Thus, when assumption **A9** holds (e.g. when ‘type’ can be viewed as physical capital), complementarity in the underlying technology gives a comparative advantage to the firms and might make them more likely to arise. In contrast, if assumption **A10** holds (e.g. when ‘type’ represents human capital), the firms are at relative disadvantage, which might make individual partnerships more likely to appear. This observation could potentially explain why partnerships have been prominent in human-capital-intensive professional services (such as law, accounting, consulting, etc.), while the corporate form or production dominates across physical-capital-intensive industries, such as manufacturing.³² Of course, the above intuition is only suggestive. To formally support it, one needs to develop a rigorous model of organizational structure, in which large firms and small partnerships could coexist. This analysis, however, is outside the scope of our paper and remains

³²Levin and Tadelis (2008) offer an alternative explanation for this observation, which relies on the differences in the partnerships’ and firms’ objectives (total profit versus profit per partner) and the choice of employees / partners quality in the environment, where prospective clients have imperfect information about the quality of the final product (and there are no informational frictions between the partners or employees).

an open question for future research.

6 Discussion

In this section we discuss how our results might change under some alternative modeling choices:

Heterogeneity in costs: It is crucial for our results that the agents are heterogenous with respect to their productivities. If they were heterogenous only with respect to costs, then the frequency of inefficient punishments in a given match would not vary with types, and therefore, we would not get matching predictions. On the other hand, adding differing costs (declining in type) on top of differing productivities would reinforce our results when effort and type are complements and weaken them when effort and type are substitutes. This is because, the size of the inefficient punishment required to satisfy incentive compatibility is proportional to the effort cost. Therefore, in the former case, the size of the punishment will be decreasing even faster with types and in the latter it would be increasing slower (and maybe even decreasing) with types.

Balanced-budget contracts, renegotiation proofness: If we rule out the assumption of commitment to discarding of resources in our main model, it is not possible for a matched pair to write a contract that implements effort by both partners. This is due to our assumption **A5**. In this case, in every match one of the partners will work and one will remain idle. Therefore, the sorting patterns will mimic those in the case of “imprecise monitoring” analyzed in Section 3.

Similar results would be obtained in the repeated model described in Section 4, if attention is restricted to the payoffs generated by “renegotiation proof” equilibria of the repeated partnership game played among matched partners - that is, if in that repeated game, continuation values are restricted to come from the Pareto frontier. This is because in the repeated game, moving to inefficient equilibria after low output is analogous to discarding of resources after low output in the static model. Renegotiation-proofness rules out such punishments.

Non-transferability: It may be of interest to know how our results would change if the output of a partnership was not divisible and each partner would automatically receive half

of the output. In this case, the static partnership game described in the main model turns into a typical prisoner's dilemma. Therefore, it has a unique equilibrium in which no partner exerts effort and hence matching patterns are immaterial for maximization of surplus. On the other hand, if the interaction between the partners is modeled as a repeated prisoner's dilemma game, there is a non-degenerate set of payoffs that can be obtained by a matched pair. In this case, even though the utility is not fully transferable, it is possible to transfer utility between agents via equilibria prescribing asymmetric effort choices. Therefore, this model would fall into the realm of models studied in Legros and Newman (2007), where utility can be transferred at a rate different than 1. With some additional simplifying restrictions on the modeling environment, it can be shown that such modification of the model favors positive assortative matching.³³ The intuition for this result is simple and is based on Legros and Newman (2007): the only way an agent transfers utility to his partner is via working harder and allowing the partner to shirk on some occasions. Therefore, it is harder for a lower type to transfer utility to his partner. Hence, lower types are unable to bid for higher type partners as efficiently as the higher type partners would. This breaks down the rationale for negative assortative matching even in the cases where total surplus would be maximized under such a matching.

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³³We make this statement based on a canonical example where the effort and type are complements and there are only two types. Therefore, in the case of divisible output, the equilibria under this technology would exhibit negative assortative matching, regardless of whether there is precise or imprecise monitoring. We are able to show that, when output is indivisible, in the case of imprecise monitoring, where only one of the partners in each match exerts effort, the equilibrium matchings are positive assortative. For the case where the monitoring technology is precise enough, we have numerical results suggesting that positive assortative matching would obtain in equilibrium.

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7 Appendix

7.1 Relationship between assumption A5 and the necessary and sufficient condition for inefficiency in partnerships provided in Legros and Matsushima (1991).

Theorem 2 in Legros and Matsushima (1991) states that the solution to the partnership problem satisfy budget balancing if and only if an index of the likelihood of deviations (measured as the average gain from deviations divided by a measure of the closeness of distributions induced by deviations) is bounded from above for all possible deviations. In our model, since there are only two output outcomes, $p(m, n) \in (0, 1)$ and $q_i(m, n) \in (0, 1)$, $i = 1, 2$ for all $m, n \in \mathcal{M}$, there exist (possibly, mixed) deviation strategies which induce identical distributions over output outcomes. Namely, there exist $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$, such that

$$(21) \quad (1 - \alpha_1)p(m, n) + \alpha_1 q_2(m, n) = (1 - \alpha_2)p(m, n) + \alpha_2 q_1(m, n),$$

where α_1 and α_2 are the probabilities with which partner 1 and partner 2 deviate.³⁴ Thus, the index of the likelihood of deviations is bounded from above for all possible strategies if and only if for all deviations (α_1, α_2) that satisfy (21), the total utility gain from these deviations is negative. Such utility gain is measured as³⁵

$$\Delta = \alpha_1 \left(c - (p(m, n) - q_2(m, n)) \frac{\bar{y} - y}{2} \right) + \alpha_2 \left(c - (p(m, n) - q_1(m, n)) \frac{\bar{y} - y}{2} \right).$$

Using (21), it can be rearranged as

$$\Delta = \alpha_1 (p(m, n) - q_2(m, n)) \left(\frac{c}{p(m, n) - q_2(m, n)} + \frac{c}{p(m, n) - q_1(m, n)} - (\bar{y} - y) \right),$$

which is negative if and only if assumption **A5** is violated.

³⁴Equivalently, we can say that the *pairwise identifiability condition* (due to Fudenberg et al. (1995)) is violated in our model.

³⁵Here we assume, without loss of generality, that each partner is entitled to one half of the realized output.

7.2 Alternative model: the repeated partnership game

7.2.1 Description of the game

Assume that in the stage game every partner is entitled to half of the realized output $\frac{y}{2}$. After the realization of the output, the partners simultaneously make transfers $t_1(y)$ and $t_2(y)$ to each other. Here, $t_i(\cdot)$ represents the net transfer that player i receives, and hence the total payment to player i is $\frac{y}{2} + t_i(y)$. In contrast to the model in the main text, assume that the partners cannot commit to money burning. That is, the sum of the ex-post payments to each of the players is equal to the total output (no money burning), i.e. $t_1(y) + t_2(y) = y$. In addition, suppose that agents have limited liability, in the sense that transfers cannot exceed the total output a partner is entitled to in one period, i.e. $t_i(y) \in [-\frac{y}{2}, \frac{y}{2}]$.³⁶ Throughout, t refers to a profile of transfers and e refers to an effort profile (e_1, e_2) . For convenience we use the notation $\bar{t}_i = t_i(\bar{y})$ and $\underline{t}_i = t_i(\underline{y})$, $i = 1, 2$.

In the repeated game, each player discounts the future with a common discount factor δ . Each period, the realized output y and the transfers t are publicly observable while the choice of effort by each player is not. The public outcome $h(\tau)$ in period τ consists of the realized output $y(\tau) \in Y$ and transfers $t(\tau) \in \mathbb{R}^4$. A public history of length τ is therefore $h^\tau = (h(1), \dots, h(\tau))$. Let \mathcal{H}^τ represent the set of all public histories of length τ and $\mathcal{H} = \bigcup_{\tau=1}^{\infty} \mathcal{H}^\tau \cup \{h_0\}$ represent the set of all public histories. Here h_0 is the null history. A pure public strategy for player i is a map $\sigma_i : \mathcal{H} \rightarrow \{E, S\} \times \mathbb{R}^2$ that maps each public history to a stage game strategy of player i , consisting of effort choice and output-contingent transfers. We focus on the pure strategy public perfect equilibria (PSPPE) of this game.³⁷ A PSPPE is a public strategy profile $\sigma = (\sigma_1, \sigma_2)$ such that σ_1 is a best response to σ_2 and vice versa.

Let $W_{mn}(\delta)$ represent the set of PSPPE payoff vectors of the repeated partnership game

³⁶In fact, the assumption we need is that the enforceable transfers are bounded. We use this particular bound because it seems to be a natural one used in the literature. Boundedness guarantees the existence of a solution for some of the intermediate optimization problems we solve in what follows.

³⁷Allowing for mixed strategies complicates the characterization of the equilibrium payoff set because if $q_i > p/2$ then a randomization between effort 0 and 1 eases the incentive constraints of partner $-i$, while reducing the expected output. How these effects balance out depends on the parameters.

for discount factor δ when the partners have types m and n , respectively. Also define the Pareto frontier of $W_{mn}(\delta)$ by

$$\mathcal{W}_{mn}^\delta(v) = \sup\{w | \exists v' \geq v \text{ such that } (v', w) \in W_{mn}(\delta)\}$$

Let $W_{mn} = \lim_{\delta \rightarrow 1} W_{mn}(\delta)$ be the limit set of the PSPPE payoff vectors, where the limit is with respect to the Hausdorff distance. Finally, define the Pareto frontier of W_{mn} by

$$\mathcal{W}_{mn}(v) = \sup\{w | \exists v' \geq v \text{ such that } (v', w) \in W_{mn}\}$$

That is, $\mathcal{W}_{mn}(v)$ is the maximum payoff that player 2 can get among equilibria where player 1's payoff is at least v as $\delta \rightarrow 1$.

7.2.2 The matching game: additional definitions

In the definition of an equilibrium given in the main text, only the definition of a feasible payoff vector is adjusted: the payoff pair (v, w) is feasible for a pair of agents i, j if $w \leq \mathcal{W}_{\kappa(i)\kappa(j)}^\delta(v)$. Given this definition of feasibility, the equilibrium definition is unchanged.

In environments where the sum of payoffs to a matched pair may vary along the Pareto frontier of achievable payoffs (non-transferable utility case), Legros and Newman (2007) introduce the following generalization of increasing (decreasing) differences property. They also establish that, in such an environment, this property is sufficient for all equilibrium matchings to be positive (negative) assortative.

Definition 1 *A type-dependent utility possibilities frontier $\mathcal{W}_{mn}^\delta : \mathbb{R} \rightarrow \mathbb{R}$ satisfies generalized increasing (decreasing) differences if for all $m > m'$ and $n > n'$*

$$\forall v, v' : \mathcal{W}_{m,n'}^\delta(v) = \mathcal{W}_{m',n'}^\delta(v') \Rightarrow \mathcal{W}_{m,n}^\delta(v) > \mathcal{W}_{m',n}^\delta(v')$$

7.2.3 Characterization of equilibrium payoffs of the partnership game

Once a matching is formed, the partners in a match play a repeated partnership game described in the previous section. As is well-known in this setting, the set of equilibrium payoff vectors is difficult to characterize. However, it is possible to bound the equilibrium payoff set using techniques introduced in Fudenberg et al. (1994). The following proposition

is a direct application of Fudenberg et al. (1994)'s result to our setting and establishes that the Pareto frontier of the PSPPE payoffs of the repeated partnership game converges to the Pareto frontier of the static partnership game discussed in the main text:

Proposition 5 *Define*

$$(22) \quad \eta(m, n) = (\bar{y} - \underline{y}) - \frac{c}{p(m, n) - q_2(m, n)} - \frac{c}{p(m, n) - q_1(m, n)}$$

For any $m \geq n$, let $\mathcal{W}_{mn}(v) = -v + S(m, n)$ where

$$(23) \quad S(m, n) = p(m, n)\bar{y} + (1 - p(m, n))\underline{y} - 2c - \min \{ (1 - p(m, n))\eta(m, n), (p(m, n) - q_1(m, n))(\bar{y} - \underline{y}) + c \}$$

For any $\varepsilon > 0$ there exists $\bar{\delta}_{mn}(\varepsilon) < 1$ such that for any $\delta > \bar{\delta}_{mn}(\varepsilon)$ and for all v :

$$\mathcal{W}_{mn}^\delta(v) \in (\mathcal{W}_{mn}(v) - \varepsilon, \mathcal{W}_{mn}(v)]$$

Proposition 5 introduces the Pareto frontier ($\mathcal{W}(\cdot)$) of a set that bounds the equilibrium payoff vectors, which turns out to coincide with the Pareto frontier of the feasible payoffs in the static model from the main text, and states that the true Pareto frontier of the equilibrium payoff vectors converges to this bound as the discount factor δ approaches 1.

Proof: We start by deriving the limit Pareto frontier (Steps 1-5), and then establish convergence, which involves minor modification of the Fudenberg et al. (1994) result (Step 6).

Step 1: Methodology of characterizing the limit Pareto frontier:

In the Fudenberg et al. (1994) characterization, the bounding set that is shown to be the limit of the set of equilibrium value vectors is the intersection of the largest half-spaces in each direction whose boundary values can be decomposed on these hyperplanes. To be more specific, a few definitions should be introduced.

- A half space $H(\lambda, k)$ with direction $(\lambda, 1 - \lambda)$ and level k is the set

$$H(\lambda, k) = \{v \in \mathbb{R}^2 \mid \lambda v_1 + (1 - \lambda)v_2 \leq k\}.$$

For brevity, define $u_i(e, t) = E\{\frac{y}{2} + t_i(y)|e\} - C(e_i)$ where $E\{\cdot|e\}$ represents the expectation taken with respect to y using the distribution induced by the effort profile e .

- A value vector $v = (v_1, v_2)$ is *decomposable on a set* $W \in \mathbb{R}^2$ if there exists an effort profile e , transfers t and continuation value vectors $\gamma(y) \in W$ for each $y \in \{\bar{y}, \underline{y}\}$ such that

$$(PK) \quad v_i = (1 - \delta)u_i(e, t) + \delta E\{\gamma(y)|e\}$$

$$(IC) \quad v_i \geq (1 - \delta)u_i((e'_i, e_{-i}), t) + \delta E\{\gamma_i(y)|e'_i, e_{-i}\} \text{ for any } e'_i \in \{0, 1\}$$

The first condition is the promise keeping condition: it guarantees that the current payoff and the expected continuation payoff average to v . The second condition is the standard incentive compatibility condition. Strictly speaking, the definition should also include the conditions stipulating that the transfers are also incentive compatible. That is,

$$(24) \quad \forall y : (1 - \delta)t_i(y) + \delta\gamma_i(y) \geq 0$$

The constraint takes this form because the transfers are observable and deviations can be punished by switching to the worst equilibrium with payoffs $(0, 0)$. Notice that as $\delta \rightarrow 1$, (24) becomes $\gamma_i(y) \geq 0$. Since we are characterizing the limit case, in what follows, we ignore this constraint keeping in mind that $v, w \geq 0$.

Then, *the largest half-space in direction* λ *whose boundary values can be decomposed on itself by effort profile* e *and transfers* t *is* $H(\lambda, k^*(\lambda, e, t))$ where $k^*(\lambda, e, t)$ is characterized by the following linear programming problem:

$$(25) \quad \begin{aligned} k^*(\lambda, e, t) &= \max_x \quad \lambda[u_1(e, t) + E_y\{x_1(y)|e\}] + (1 - \lambda)[u_1(e, t) + E_y\{x_2(y)|e\}] \\ \text{s.t.:} \quad &u_1(e, t) + E_y\{x_1(y)|e\} \geq u_i(e'_i, e_{-i}, t) + E_y\{x_i(y)|e'_i, e_{-i}\} \\ &\lambda x_1(y) + (1 - \lambda)x_2(y) \leq 0 \end{aligned}$$

This can be seen by noting that if the continuation values $x_i(y)$ are obtained via the normalization $x_i(y) = (\gamma_i(y) - v) \frac{\delta}{1 - \delta}$ from unnormalized continuation values $\gamma_i(y)$, the first constraint is equivalent to the condition *(IC)*, the objective function is nothing but $\lambda v_1 + (1 - \lambda)v_2$ for some $v = (v_1, v_2)$ for which *(PK)* is satisfied. Finally, the last constraint guarantees

that the unnormalized continuation values γ come from the hyperplane $H(\lambda, k^*(\lambda, e, t))$.

Define

$$k^*(\lambda) = \max_{e,t} k^*(\lambda, e, t) = \max_e \max_t k^*(\lambda, e, t).$$

Therefore, $H(\lambda, k^*(\lambda))$ is *the largest half space in direction λ* .

Fudenberg et al. (1994) show that the set of equilibrium payoffs is bounded by

$$W = \bigcap_{\lambda} H(\lambda, k^*(\lambda))$$

Obviously, W includes the set of PSPPE payoff vectors $W(\delta)$, and below we also show (using FLM methodology) that $W(\delta)$ converges to W as $\delta \rightarrow 1$.

Steps 2-5 below are devoted to characterizing the set W for our repeated partnership game. We characterize $k^*(\lambda, e)$ for each possible combination of efforts³⁸(Steps 2 and 3), then find $k^*(\lambda) = \max_e k^*(\lambda, e)$ (Step 4), and, finally, characterize W and its Pareto frontier $\mathcal{W}(v)$ (Step 5). Since all the analysis is carried out for a given partnership (m, n) , we drop reference to the partners' types throughout.

Step 2: Characterizing $k^*(\lambda, EE,)$

First, we find $k^*(\lambda, EE, t)$ for all t and then maximize it with respect to t to find $k^*(\lambda, EE)$. The linear programming problem (25) described in the previous section becomes:

$$\begin{aligned} k^*(\lambda, EE, t) = \max_{x_1, x_2} \lambda & \left(\frac{p\bar{y} + (1-p)\underline{y}}{2} - c + p\bar{t}_1 + (1-p)\underline{t}_1 + px_1(\bar{y}) + (1-p)x_1(\underline{y}) \right) + \dots \\ & \dots + (1-\lambda) \left(\frac{p\bar{y} + (1-p)\underline{y}}{2} - c + p\bar{t}_2 + (1-p)\underline{t}_2 + px_2(\bar{y}) + (1-p)x_2(\underline{y}) \right) \\ \text{subject to: } & \frac{1}{p-q_2} \left(c - \frac{(p-q_2)(\bar{y}-\underline{y})}{2} \right) - (\bar{t}_1 - \underline{t}_1) \leq x_1(\bar{y}) - x_1(\underline{y}) \\ & \frac{1}{p-q_1} \left(c - \frac{(p-q_1)(\bar{y}-\underline{y})}{2} \right) - (\bar{t}_2 - \underline{t}_2) \leq x_2(\bar{y}) - x_2(\underline{y}) \\ & \lambda x_1(\underline{y}) + (1-\lambda)x_2(\underline{y}) \leq 0 \quad \text{for all } y \in Y \end{aligned}$$

Denote the left hand side of the IC constraint for agent i by L_i . That is,

$$L_1 = \frac{1}{p-q_2} \left(c_1 - \frac{(p-q_2)(\bar{y}-\underline{y})}{2} \right) - (\bar{t}_1 - \underline{t}_1).$$

³⁸In fact, we omit the characterization of $k^*(\lambda, SS)$ noting that it is trivially equal to 0.

$$L_2 = \frac{1}{p - q_1} \left(c_n - \frac{(p - q_2)(\bar{y} - \underline{y})}{2} \right) - (\bar{t}_2 - \underline{t}_2).$$

To characterize the solution to this problem, it is convenient to distinguish between two separate cases,

$$(26) \quad \lambda L_1 + (1 - \lambda)L_2 \leq 0$$

and

$$(27) \quad \lambda L_1 + (1 - \lambda)L_2 > 0.$$

If λ and t satisfy the first inequality then it is possible to choose $x_1(\cdot), x_2(\cdot)$ in such a way that $\lambda x_1(\bar{y}) + (1 - \lambda)x_2(\bar{y}) = \lambda x_1(\underline{y}) + (1 - \lambda)x_2(\underline{y}) = 0$ and incentive constraints are satisfied.³⁹ Therefore, in this case

$$(28) \quad k^*(\lambda, EE, t) = \lambda \left(\frac{p\bar{y} + (1 - p)\underline{y}}{2} - c + p\bar{t}_1 + (1 - p)\underline{t}_1 \right) + (1 - \lambda) \left(\frac{p\bar{y} + (1 - p)\underline{y}}{2} - c + p\bar{t}_2 + (1 - p)\underline{t}_2 \right)$$

Thus (26) implies that orthogonal implementation is possible.

On the other hand, if λ and t are such that (27) holds, then

$$\lambda x_1(\underline{y}) + (1 - \lambda)x_2(\underline{y}) < \lambda x_1(\bar{y}) + (1 - \lambda)x_2(\bar{y})$$

and it would be optimal to choose a combination of $x_1(\bar{y})$ and $x_2(\bar{y})$ that satisfies $\lambda x_1(\bar{y}) + (1 - \lambda)x_2(\bar{y}) = 0$. Obviously, $\lambda x_1(\underline{y}) + (1 - \lambda)x_2(\underline{y}) = 0$ is not feasible any more, and in the optimal solution both incentive constraints must bind. Therefore

$$(29) \quad k^*(\lambda, EE, t) = \lambda \left[\frac{p\bar{y} + (1 - p)\underline{y}}{2} - c + p\bar{t}_1 + (1 - p)\underline{t}_1 - (1 - p)L_1 \right] + \dots \\ \dots (1 - \lambda) \left[\frac{p\bar{y} + (1 - p)\underline{y}}{2} - c + p\bar{t}_2 + (1 - p)\underline{t}_2 - (1 - p)L_2 \right].$$

Note that if (26) holds with equality then both incentive constraints must bind and therefore equations (29) and (28) deliver the same value.

The next step is to choose the transfer profile $t^*(\lambda)$ that maximizes the level of the hyperplane in the direction λ when EE is the effort profile. This is done in the following Lemma.

³⁹If (26) holds with strict inequality, at least one of the incentive constraints will be slack.

Lemma 1 Define for all y :

$$(30) \quad t_1^*(y) = -t_2^*(y) = \begin{cases} \frac{y}{2} & \text{if } \lambda \geq \frac{1}{2} \\ -\frac{y}{2} & \text{if } \lambda < \frac{1}{2} \end{cases}$$

Then $t^* \in \mathbf{argmax}_t k^*(\lambda, EE, t)$.

Proof: The result follows from observing that t^* defined in (1) maximizes (28) and minimizes the left hand side of (26). \square

This Lemma implies that for large δ and when v, w is on the frontier, one of the agents receives all of the current output if both agents work. Therefore, $k^*(\lambda, EE)$ can be expressed as follows:

$$(31) \quad k^*(\lambda, EE) = \begin{cases} \lambda(p\bar{y} + (1-p)\underline{y} - c) + (1-\lambda)(-c) - (1-p) \max\{0, \eta_1(\lambda)\} & \text{if } \lambda > \frac{1}{2} \\ \lambda(-c) + (1-\lambda)(p\bar{y} + (1-p)\underline{y} - c) - (1-p) \max\{0, \eta_2(\lambda)\} & \text{if } \lambda \leq \frac{1}{2} \end{cases}$$

where

$$\eta_1(\lambda) = \lambda \frac{c - (p - q_2)(\bar{y} - \underline{y})}{p - q_2} + (1 - \lambda) \frac{c}{p - q_1}$$

and

$$\eta_2(\lambda) = \lambda \frac{c}{p - q_2} + (1 - \lambda) \frac{c - (p - q_1)(\bar{y} - \underline{y})}{p - q_1}$$

Note that $\eta_1(\lambda)$ and $\eta_2(\lambda)$ are the values of $\lambda L_1 + (1-\lambda)L_2$ evaluated at the corresponding optimal t 's. The term $\max\{0, \eta_1(\lambda)\}$ in equation (31) becomes positive when the condition (27) holds: ICs are binding and orthogonal implementation is not possible.

Correspondingly, the hyperplane associated with the optimal transfer schedule for $\lambda \geq \frac{1}{2}$ [the hyperplane $\lambda v + (1-\lambda)w = k^*(\lambda, EE)$] passes through either A_1 or B_1 defined below—namely the one which delivers a lower level in this direction λ .⁴⁰

$$(32) \quad \begin{aligned} A_1 : & \quad (p\bar{y} + (1-p)\underline{y} - c, -c) \\ B_1 : & \quad \left(p\bar{y} + (1-p)\underline{y} - c - \frac{1-p}{p-q_2}(c - (p-q_2)(\bar{y} - \underline{y})), -c - \frac{1-p}{p-q_1}c \right) \end{aligned}$$

⁴⁰By this we mean, the inner product $(\lambda, 1-\lambda) \times (v, w)$ is minimized.

For $\lambda \leq \frac{1}{2}$ the corresponding hyperplane passes through the lower one of the following two points:

$$(33) \quad \begin{aligned} A_2 : & \quad (-c, p\bar{y} + (1-p)\underline{y} - c) \\ B_2 : & \quad \left(-c - \frac{1-p}{p-q_2}c, p\bar{y} + (1-p)\underline{y} - c - \frac{1-p}{p-q_1}(c - (p-q_1)(\bar{y} - \underline{y})) \right) \end{aligned}$$

Step 3: Characterizing $k^*(\lambda, ES)$ and $k^*(\lambda, SE)$

For the effort profile ES and transfers t , the linear programming problem (25) can be written as

$$(34) \quad \begin{aligned} k^*(\lambda, ES, t) = \mathbf{argmax}_x & \quad \lambda[q_1(\bar{y}/2 + \bar{t}_1) + (1-q_1)(\underline{y}/2 + \underline{t}_1) - c + q_1x_1(\bar{y}) + (1-q_1)x_1(\underline{y})] + \dots \\ & \quad \dots(1-\lambda)[q_1(\bar{y} + \bar{t}_2) + (1-q_1)(\underline{y}/2 + \underline{t}_2) + q_2x_2(\bar{y}) + (1-q_2)x_2(\underline{y})] \\ \text{subject to: } & \quad x_1(\bar{y}) - x_1(\underline{y}) \geq \frac{1}{q_1} \left(c - \frac{q_1(\bar{y} - \underline{y})}{2} \right) - (\bar{t}_1 - \underline{t}_1) \\ & \quad x_2(\bar{y}) - x_2(\underline{y}) \leq \frac{1}{p-q_1} \left(c - \frac{(p-q_1)(\bar{y} - \underline{y})}{2} \right) - (\bar{t}_2 - \underline{t}_2) \\ & \quad \lambda x_1(\underline{y}) + (1-\lambda)x_2(\underline{y}) \leq 0; \quad y \in \{\underline{y}, \bar{y}\} \end{aligned}$$

If there were no incentive constraints, it would always be possible to enforce (ES, t) orthogonally, which would deliver

$$(35) \quad k^*(\lambda, ES, t) = \lambda[-c + q_1(\bar{y} + \bar{t}_1) + (1-q_1)(\underline{y} + \underline{t}_1)] + (1-\lambda)[q_1(\bar{y} + \bar{t}_1) + (1-q_1)(\underline{y} + \underline{t}_2)].$$

Since the incentive constraints in (34) bound $x_1(\bar{y}) - x_1(\underline{y})$ from below and $x_2(\bar{y}) - x_2(\underline{y})$ from above, it is always possible to choose such x that $\lambda x_1(\underline{y}) + (1-\lambda)x_2(\underline{y}) = 0$ for any y and both incentive constraints are satisfied.⁴¹ Thus (35) is also a solution to the linear program (34).

To maximize $k^*(\lambda, ES, t)$ with respect to t we need to set $\bar{t}_1 = -\bar{t}_2 = -\frac{\bar{y}}{2}$ and $\underline{t}_1 = -\underline{t}_2 = -\frac{\underline{y}}{2}$ if $\lambda < \frac{1}{2}$ and similarly $\bar{t}_1 = -\bar{t}_2 = \frac{\bar{y}}{2}$ and $\underline{t}_1 = -\underline{t}_2 = \frac{\underline{y}}{2}$ otherwise. Therefore,

$$k^*(\lambda, ES) = \begin{cases} \lambda[q_1\bar{y} + (1-q_1)\underline{y} - c] & \text{if } \lambda \geq \frac{1}{2} \\ \lambda(-c) + (1-\lambda)[q_1\bar{y} + (1-q_1)\underline{y}] & \text{otherwise} \end{cases}.$$

⁴¹We only need to make sure that $x_1(\bar{y}) - x_1(\underline{y})$ and $x_2(\bar{y}) - x_2(\underline{y})$ are sufficiently far away from each other.

The level of the largest half-space in direction λ decomposed on itself by SE can be straightforwardly determined in a similar way:

$$k^*(\lambda, SE) = \begin{cases} \lambda[q_2\bar{y} + (1 - q_2)\underline{y}] + (1 - \lambda)(-c) & \text{if } \lambda \geq \frac{1}{2} \\ (1 - \lambda)[q_2\bar{y} + (1 - q_2)\underline{y} - c] & \text{otherwise} \end{cases}.$$

Step 4: Characterizing $k^*(\lambda)$

For each λ the level of largest half space in the direction λ is found as

$$(36) \quad k^*(\lambda) = \max\{k^*(\lambda, EE), k^*(\lambda, ES), k^*(\lambda, SE)\}$$

It is convenient to first characterize $\max\{k^*(\lambda, ES), k^*(\lambda, SE)\}$ and then compare it with $k^*(\lambda, EE)$.

For $\lambda \geq \frac{1}{2}$ the hyperplane $\lambda v + (1 - \lambda)w = k^*(\lambda, ES)$ passes through D_1 and hyperplane $\lambda v + (1 - \lambda)w = k^*(\lambda, SE)$ passes through G_1 defined as follows:

$$(37) \quad \begin{aligned} D_1 : & \quad (q_1\bar{y} + (1 - q_1)\underline{y} - c, 0) \\ G_1 : & \quad (q_2\bar{y} + (1 - q_2)\underline{y}, -c) \end{aligned}$$

For $\lambda \leq \frac{1}{2}$ the corresponding hyperplanes pass through D_2 and G_2 :

$$(38) \quad \begin{aligned} D_2 : & \quad (-c, q_1\bar{y} + (1 - q_1)\underline{y}) \\ G_2 : & \quad (0, q_2\bar{y} + (1 - q_2)\underline{y} - c) \end{aligned}$$

It is easy to see that for $m > n$ the points G_1, G_2 lie below the line connecting the points D_1 and D_2 .⁴² Therefore, for $\lambda \geq \frac{1}{2}$, the hyperplane passing through D_1 should be chosen and for $\lambda \leq \frac{1}{2}$ the hyperplane passing through D_2 should be chosen. This is intuitive because it implies that whenever only one of the agents works it is the more efficient one.

In the light of this discussion, we get that whenever $m > n$:

$$k^*(\lambda) = \max\{k^*(\lambda, EE), k^*(\lambda, ES)\}.$$

Step 5: Characterization of $\mathcal{W}(\cdot)$

Recall that

$$\mathcal{W} = \bigcap_{\lambda} \{(v, w) | \lambda v + (1 - \lambda)w \leq k^*(\lambda)\}$$

⁴²To see this note that the line connecting D_1 and D_2 has slope -1 and level $f(1, 0) - c$ while the lines through G_1 and G_2 with slope -1 have levels $f(0, 1) - c < f(1, 0) - c$.

Proposition 6 *Define*

$$\eta = \frac{c - (p - q_2)(\bar{y} - \underline{y})/2}{p - q_2} + \frac{c - (p - q_1)(\bar{y} - \underline{y})/2}{p - q_1}$$

Then,

$$\mathcal{W}(v) = -v + p\bar{y} - 2c - \min\{2(1 - p)\eta, (p - q_1)(\bar{y} - \underline{y}) + c\}$$

Proof: First, notice that $\overline{A_1A_2}$, $\overline{B_1B_2}$ and $\overline{D_1D_2}$ all have slopes -1. Also, all points $A_1, A_2, B_1, B_2, D_1, D_2$ lie outside of the positive orthant. Next, observe that $\overline{A_1A_2}$ lies above $\overline{B_1B_2}$ by assumption **A5**, and by assumption **A1**, $\overline{A_1A_2}$ lies above $\overline{D_1D_2}$. Finally, observe that $p\bar{y} + (1 - p)\underline{y} - 2c - 2\eta$ and $q_1\bar{y} + (1 - q_1)\underline{y} - c$ are the levels of $\overline{B_1B_2}$ and $\overline{D_1D_2}$, respectively. \square

Step 6: Convergence

The following proposition reproduces the result of Fudenberg et al. (1994) that \mathcal{W} is the limit of the equilibrium payoff set as δ converges to 1 and the proof of Proposition 5 follows directly from it.

Proposition 7 (*Fudenberg et al. (1994)*) *Let $V \subset \text{int}W$ be smooth⁴³ and convex. Then there exists $\bar{\delta}$ such that for any $\delta > \bar{\delta}$, $V \subset W(\delta)$.*

Proof: We note that the Pareto frontier of the repeated partnership game is obtained using $(\bar{t}_i, \underline{t}_i) \in \{-\bar{y}/2, \bar{y}/2\} \times \{\underline{y}/2, -\underline{y}/2\}$. That is, restricting attention to equilibria that use only these transfers does not shrink the equilibrium set. Therefore, the proof directly follows from Fudenberg et al. (1994). \square

7.2.4 Matching patterns: proof of Proposition 3

Proposition 3 follows from Proposition 5 above, Proposition 2 in the main text and the following result which proves that if the limit frontier $\mathcal{W}_{mn}(v)$ satisfies increasing (decreasing) differences, then for large δ , $\mathcal{W}_{mn}^\delta(v)$ also does.

⁴³A smooth set is closed, with non-empty interior and its boundary is twice continuously differentiable.

Lemma 2 *Assume $S(m, n)$ exhibits (decreasing) differences. Then there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, $\mathcal{W}_{mn}^\delta(\cdot)$ exhibits generalized increasing (decreasing) differences.*

Proof: Let $T = \{\kappa(i) | i \in N\}$. That is T is the set of types of the N agents in the economy. First assume increasing differences, i.e.

$$(S(m, n) - S(m, n')) - (S(m', n) - S(m', n')) > 0$$

Let $\varepsilon = \inf\{S(m, n) + S(m', n') - S(m, n') - S(m', n) | m > m', n > n' \in T\}$ and $\varepsilon^* = \frac{1}{4}\varepsilon$. For each m, n there exists $\delta_{mn}(\varepsilon^*)$, such that for all $\delta > \delta_{mn}(\varepsilon^*)$, $\sup_v |\mathcal{W}^\delta(v) - \mathcal{W}(v)| < \varepsilon^*$. Now, let $\bar{\delta} = \max\{\bar{\delta}_{mn}(\varepsilon^*) | m, n \in T\}$. Since T is finite, $\varepsilon^* > 0$ and $\bar{\delta} < 1$.

Take $\delta > \bar{\delta}$, $m > m', n > n'$ and v, v' such that

$$\mathcal{W}_{mn'}^\delta(v) = \mathcal{W}_{m'n'}^\delta(v')$$

Then,

$$\mathcal{W}_{mn'}(v) - \mathcal{W}_{m'n'}(v') > -2\varepsilon^*$$

Moreover,

$$\mathcal{W}_{mn}^\delta(v) - \mathcal{W}_{m'n}^\delta(v') > \mathcal{W}_{mn}(v) - \mathcal{W}_{m'n}(v') - 2\varepsilon^*$$

Also,

$$(\mathcal{W}_{mn}(v) - \mathcal{W}_{m'n}(v')) - (\mathcal{W}_{mn'}(v) - \mathcal{W}_{m'n'}(v')) = (S(m, n) - S(m'n)) - (S(m, n') - S(m', n')) \geq 4\varepsilon^*$$

The last three inequalities together imply

$$\mathcal{W}_{mn}^\delta(v) - \mathcal{W}_{m'n}^\delta(v') > 0$$

which establishes generalized increasing differences. The proof for generalized decreasing differences is analogous. \square